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AN APPROACH TO THE RADON-NIKODYM THEOREM

By V. VIKRAMAN

INTRODUCTION. The aim of this paper is the study of the system (μ) of all bounded signed measures on a Boolean σ-algebra B of subsets of an abstract space Ω. In the treatment in usual treatises, the partial order and the lattice structure of the system (μ) of measures is ignored. Here, however, our main purpose is to show how the usual properties of measures, in particular the Jordan, Hahn and Lebesgue decompositions arise in a direct and natural manner from the lattice structures of the system of all bounded signed measures. We also show that the notions of absolute continuity and singularity of measures are closely connected with the lattice structure of (μ) and we base our treatment of them on their lattice-theoretic definition; we also obtain the Radon-Nikodym theorem by means of partitions arising from the limit of $(n\,\mu \wedge \nu)$ if $\mu, \nu \geqslant 0$ and $\nu \ll \mu$. The special features believed to be new* in this paper are:

(i) The explicit definition of the lattice sum and product of measures as certain upper and lower bounds for all possible partitions and the fact that these are actually attained in the case of a finite number of measures in certain partitions which I have called 'extremal' partitions and the fact that these extremal partitions lie behind the usual decompositions.

(ii) The lattice theoretic treatment of the relations of singularity and absolute continuity between measures.

(iii) It becomes apparent from our treatment of absolute continuity that essentially the only measures absolutely continuous with respect to any measure μ are its truncations; in other words any measure $\nu \ll \mu$ can be approximated to any desired

*It has been pointed out by the referee that these notions are already familiar and my attention has been drawn to John Von Neumann's *Functional operators,* Vol. 1 pp. 180-181, and P. R. Halmos's *Introduction to Hilbert space,* p 79.

extent by linear combinations of truncations of μ. It is this fact again which leads to the Radon-Nikodym theorem.

For certain proofs I have referred to Birkhoff: *Lattice theory* (1949). The other relevant references are Halmos: *Measure theory*, and Saks: *The theory of the integral.*

I am grateful to Dr. R. Vaidyanathaswamy for the discussions I had with him on this problem.

I. Lattice structure of the system

1.1. The system (μ) of bounded signed measures on B can be partially ordered by the relation $\geqslant$, where $\mu_1 \geqslant \mu_2$ is defined to mean $\mu_1(X) \geqslant \mu_2(X)$ for every set $X \in B$. The zero-measure, i.e. the measure which assigns the value 0 to every set $X \in B$, is denoted by 0. A bounded signed measure μ on B is 'positive' if $\mu \geqslant 0$; μ is negative if $\mu \leqslant 0$.

In this connection we observe that if $\mu \geqslant 0$, the family of sets $\{X\}$ $(X \in B)$ for which $\mu(X) = 0$ is a σ-ideal I_μ(1). Also if $\mu, \nu \geqslant 0$ and $\mu \leqslant \nu$ then $I_\mu \supset I_\nu$.

We shall now prove* a theorem which establishes that (μ) is a lattice.

THEOREM 1. *The least upper bound and the greatest lower bound of any two bounded signed measures μ, ν exist and are bounded signed measures.*

(1) A sub-family J of sets $\in B$ is a σ-ideal if and only if $X_i \in J$, $i = 1, 2, \ldots$ imply $\bigcup_{i=1}^{\infty} X_i \in J$, and $X \in J$ and $Y \subset X$ imply $Y \in J$.

*It has been pointed out by the referee that much of the labour, here as also elsewhere in this paper can be saved if it is realized at the outset that

$$\mu \vee \nu = \mu + (\nu - \mu)^+ = \nu + (\mu - \nu)^+$$

and

$$\mu \wedge \nu = \mu - (\nu - \mu)^+ = \nu - (\mu - \nu)^+.$$

However, since the paper is only intended to be of an expository nature, it has been published in the original form with a view to preserving continuity.

Proof : We consider, for any set $X \in B$, the l.u.b $k(X)$ of $\{\mu(X_1) + \nu(X_2)\}$ for all possible partitions[2] $\{X_1, X_2\}$ of X. The existence of this l.u.b. is guaranteed by the hypothesis that μ, ν are bounded.

To prove that k is a measure, suppose that $\{X_i; i = 1, 2, \ldots\}$ is a denumerable partition of X. Let $(\overline{X}_i : \overline{\overline{X}}_i)$ be given partitions of X_i, $i = 1, 2, \ldots$. Then we have

$$\begin{aligned} k(X) &= \text{l. u. b. } \{\mu(Y_1) + \nu(Y_2)\}, \text{ for all partitions } \{Y_1, Y_2\} \text{ of } X. \\ &\geqslant \mu\Big(\bigcup_{i=1}^{\infty} \overline{X}_i\Big) + \nu\Big(\bigcup_{i=1}^{\infty} \overline{\overline{X}}_i\Big) \\ &= \sum_{i=1}^{\infty} \{\mu(\overline{X}_i) + \nu(\overline{\overline{X}}_i)\}. \end{aligned} \tag{1.11}$$

This is true whatever the partitions $\{\overline{X}_i, \overline{\overline{X}}_i\}$ of X. Therefore we have on replacing each term on the right side of (1.11) by its upper bound,

$$k(X) \geqslant \sum_{i=1}^{\infty} k(X_i).$$

Again, for an arbitrary partition (Y_1, Y_2) of X, we have

$$\begin{aligned} \sum_{i=1}^{\infty} k(X_i) &= \sum_{i=1}^{\infty} \text{l. u. b} \{\mu(\overline{X}_i) + \nu(\overline{\overline{X}}_i)\}, \text{ for all partitions } \{\overline{X}_i, \overline{\overline{X}}_i\} \text{ of } X_i \\ &\geqslant \sum_{i=1}^{\infty} \{\mu(Y_1 \cap X_i) + \nu(Y_2 \cap X_i)\} \\ &= \{\mu(Y_1) + \nu(Y_2)\}. \end{aligned}$$

Hence we immediately have $\sum_{i=1}^{\infty} k(X_i) \geqslant k(X)$. Thus $\sum_{i=1}^{\infty} k(X_i) = k(X)$; moreover $k(0) = 0$. Therefore k is a measure. Now, if λ is any other upper bound for μ, ν then, for any partition $\{X_1, X_2\}$ of X

$$\lambda(X) = \sum_{i=1}^{2} \lambda(X_i) \geqslant \{\mu(X_1) + \nu(X_2)\},$$

implying $\lambda(X) \geqslant k(X)$. Therefore k represents the lattice sum $(\mu \vee \nu)$ of μ and ν. The dual result is proved on similar lines. Hence the system (μ) is a lattice.

(2) A countable (i.e. finite or denumerable) partition $\{X_i; i = 1, 2 \ldots\}$ of X is a division of $X = \bigcup_{i=1}^{\infty} X_i$ into a countable number of disjoint sets $X_i \in B$.

(3) A refinement of a partition $\{X_i\}$ of a set $X \in B$ is a partition of X consisting of further divisions of the sets X_i into disjoint sets in B.

More generally we proceed to show

THEOREM 2. *The least upper bound (greatest lower bound) of any system* $\{\mu_\alpha\}$ *of bounded signed measures, order-bounded above by* μ_0 *(below by* ν_0*) exists and belongs to* (μ).

In order to prove this, we consider for any set $X \in B$, the l.u.b. $k(X)$ of $\sum_{i=1}^{\infty} \mu_i(X_i)$ (which is necessarily $< \mu_0(X)$) for all possible sequences of measures (μ_i) chosen from $\{\mu_\alpha\}$ and all possible denumerable partitions $\{X_i;\ i = 1, 2 \ldots,\}$ of X.

If (X_i) is a denumerable partition of X and (X_{ij}) $(j = 1, 2, \ldots)$ a denumerable partition of X_i, we have

$$k(X) \geqslant \sum_{ij} \mu_{ij}(X_{ij}) = \sum_i \{\sum_j \mu_{ij}(X_{ij})\}.$$

for every set of partitions (X_{ij}) of X_i and sequences (μ_{ij}), $j = 1, 2, \ldots \in \{\mu_\alpha\}$. Therefore

$$k(X) \geqslant \sum_{i=1}^{\infty} \text{l.u.b.}\{\sum_{j=1}^{\infty} \mu_{ij}(X_{ij})\} = \sum_{i=1}^{\infty} k(X_i).$$

Again, if in addition to the partitions (X_i) of X and (X_{ij}) of X_i, we consider a partition $\{\overline{X}_i\}$, $i = 1, 2 \ldots$ of X and a sequence of measures (μ_i), $i = 1, 2 \ldots$ chosen from $\{\mu_\alpha\}$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} k(X_i) &= \sum_i \text{l.u.b}\{\sum_j \mu_{ij}(X_{ij})\}.\\ &\geqslant \sum_{i=1}^{\infty}\{\sum_{j=1}^{\infty} \mu_j(\overline{X}_j \cap X_i)\}\\ &= \sum_{j=1}^{\infty} \mu_j(\overline{X}),\end{aligned}$$

interchanging order of summation. Whence

$$\sum_{i=1}^{\infty} k(X_i) \geqslant \text{l.u.b.}\{\sum_{j=1}^{\infty} \mu_j(\overline{X}_j)\} = k(X).$$

Thus $k(X) = \sum_{i=1}^{\infty} k(X_i)$; moreover $k(0) = 0$. Therefore k is a measure.

Now, if λ is any other upper bound for $\{\mu_\alpha\}$, then, for any partition $(X_i,\ i = 1, 2 \ldots)$ of X and sequence of measures $\{\mu_i\}$ from $\{\mu_\alpha\}$,

$$\lambda(X) = \sum_{i=1}^{\infty} \lambda(X_i) \geqslant \sum_{i=1}^{\infty} \mu_i(X_i)$$

implying

$$\lambda(X) \geqslant \text{l.u.b}\left\{\sum_{i=1}^{\infty} \mu_i(X_i)\right\} = k(X).$$

This proves that among all upper bounds for $\{\mu_\alpha\}$, k is the least. Therefore k represents the lattice sum $(\bigvee_\alpha \mu_\alpha)$ of the system of measures $\{\mu_\alpha\}$. The dual result is proved similarly. Hence the system (μ) of all bounded signed measures is a boundedly complete lattice, i.e. a partially ordered set in which every bounded set has a lattice sum and product.

In particular, the system (μ) is a boundedly σ-complete lattice, i.e. a lattice in which every bounded countable family has a lattice sum and product. In the case of a countable family, the actual expression for lattice sum reduces to

$$\left\{\bigvee_{i=1}^{\infty} \mu_i\right\}(X) = \text{l.u.b.}\left\{\sum_{i=1}^{\infty} \mu_i(X_i)\right\},$$

for all partitions $\{X_i;\ i = 1, 2, 3 \ldots\}$ of X.

1.2. Extremal partitions

Theorem 3. *For any set $X \in B$, there exist partitions of X for which the l.u.b. (g.l.b.) of any two measures μ, ν is actually attained.*†

Proof. We shall prove the theorem for the case of lattice sum ; proof for the dual part follows similarly.

Consider $(\mu \vee \nu)(X)$. By definition, there exists a partition $\{X_n, \overline{X - X_n}\}$ of X such that

$$\left.\begin{aligned} &\mu(X_n) + \nu(X - X_n) \geqslant (\mu \vee \nu)(X) - \frac{\epsilon}{2^n},\ n = 1, 2, \ldots \\ &\epsilon > 0 \text{ being a given number, or} \\ &\{(\mu \vee \nu) - \mu\}(X_n) + \{(\mu \vee \nu) - \nu\}(X - X_n) \leqslant \frac{\epsilon}{2^n},\ n = 1, 2, \ldots \end{aligned}\right\} \quad (1.21)$$

Write

$$A = \lim\sup_n (X_n) = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} (X_i),$$

$$(X - A) = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} (X - X_i) = \lim\inf_n (X - X_n) \subset \lim\sup_n (X - X_n)$$

$(A, \overline{X - A}$ are sets belonging to the σ-algebra $B)$.

† I believe that extremal partitions exist in the case of a countable bounded family also, but I have not arrived at a satisfactory proof.

Now

$$(\mu \vee \nu)(X) - \{\mu(A) + \nu(X - A)\}$$
$$= \{(\mu \vee \nu) - \mu\}(A) + \{(\mu \vee \nu) - \nu\}.(X - A)$$
$$\leqslant \sum_{i=k}^{\infty} [(\mu \vee \nu) - \mu](X_i) + \sum_{i=k}^{\infty} [(\mu \vee \nu) - \nu](X - X_i), \text{ for every integer } k.$$
$$< \frac{\epsilon}{2^{k-1}}, \qquad \text{by 1.21.}$$

Therefore $(\mu \vee \nu)(X) = \{\mu(A) + \nu(X - A)\}$. Thus the partition $(A, \overline{X - A})$ constitutes a maximal partition for $(\mu \vee \nu)$. This theorem is easily extended for any finite number of measures.

Corollary. *The maximal partition $(A, \overline{X - A})$ of X for $(\mu \vee \nu)$ must have the property that $\mu(S) \geqslant \nu(S)$ for every set $S \subset A$ and $\mu(T) \leqslant \nu(T)$ for every set $T \subset \overline{X - A}$;*

For, if for a set $S \subset A$, $\mu(S) < \nu(S)$ then

$$\mu(A) + \nu(X - A) < \mu(A - S) + \nu(\overline{x - A} \cup S),$$

which contradicts the maximality of the partition $(A, \overline{X - A})$.

Similarly the minimal partition $(A, \overline{X - A})$ of X for $(\mu \wedge \nu)$ is seen to have the property that $\mu(S) \leqslant \nu(S)$ for every set $S \subset A$ and $\mu(S) \geqslant \nu(T)$ for every set $T \subset (X - A)$.

Extremal partitions are uniquely determined but for sets for which $\mu = \nu$, since such sets could be shifted from one part to the other without affecting the extremal character of the partitions.

1.3. The system (μ) as a vector lattice

It is well-known that the system (μ) forms a vector space, if one defines the measures $\alpha\mu$ (α real) and $(\mu+\nu)$ by prescribing

$$\alpha\mu(X) = \alpha.\,\mu(X) \text{ and } (\mu+\nu)(X) = \mu(X) + \nu(X)$$

for every set $X \in B$. In relation to the group operation $+$, we note that the ordering relation remains invariant under all group translations, i.e. if λ, μ, ν are any measures and $\mu \geqslant \nu$ then $(\lambda + \mu) \geqslant (\lambda + \nu)$ and $(\lambda - \mu) \leqslant (\lambda - \nu)$. Thus the correspondence $\mu \to (\lambda + \mu)$ is an order-isomorphism. In other words the system (μ) as a vector space is a vector lattice. Hence any

translation $\mu \to (\lambda + \mu)$ will carry lattice sums and products into lattice sums and products respectively; and since the negative operation reverses the order, any transformation $\mu \to (\lambda - \mu)$ will carry lattice sums and products into the lattice products and sums respectively. Thus we may write down the following distributive laws for any bounded system of measures $\{\mu_\alpha\}$.

$$\begin{aligned} \lambda + \Lambda_\alpha \mu_\alpha &= \Lambda_\alpha (\lambda + \mu_\alpha), \\ \lambda + V_\alpha \mu_\alpha &= V_\alpha (\lambda + \mu_\alpha), \\ \lambda - \Lambda_\alpha \mu_\alpha &= V_\alpha (\lambda - \mu_\alpha), \\ \lambda - V_\alpha \mu_\alpha &= \Lambda_\alpha (\lambda - \mu_\alpha). \end{aligned} \tag{1.31}$$

In particular, we have from the third law, for the two measures μ, ν, with $\lambda = (\mu + \nu)$,

$$(\mu + \nu) = (\mu \, V \, \nu) + (\mu \, \Lambda \, \nu). \tag{1.32}$$

DEFINITION 1. *The positive part μ^+ of a signed measure μ is $(\mu \, V \, 0)$; the negative part μ^- of μ is $(\mu \, \Lambda \, 0)$; the absolute $|\mu|$ of μ is $(\mu^+ - \mu^-)$.*

THEOREM 4. *In the system (μ), the absolute satisfies*

(i) *If* $\mu \neq 0$, *then* $|\mu| > 0$, *while* $|0| = 0$.

(ii) $|c\mu| = |c| \, |\mu|$ *for any real c*,

(iii) $|\mu| = \{\mu \, V \, (-\mu)\}$,

(iv) $|\mu_1 + \mu_2| \leqslant |\mu_1| + |\mu_2|$,

(v) $|\mu_1 - \mu_2| = |(\mu_1 \, V \, \mu_2)| - (\mu_1 \, \Lambda \, \mu_2)$.

PROOF. (i) follows from the fact that $\mu^+ \geqslant 0$, $(-\mu^-) \geqslant 0$, from Definition 1, and $|\mu| = 0$ if and only if $\mu^+ = 0 = \mu^-$, i.e. $\mu = 0$. To prove (ii) we have from Definition 1, $|c\mu| = (c\mu)^+ - (c\mu)^-$. But for positive c, it is clear from the definition of lattice sum that $(c\mu)^+ = c(\mu^+)$ and $(c\mu)^- = c(\mu^-)$. Hence $|c\mu| = |c| \, |\mu|$. The result for negative c, now follows since $|-\mu| = |\mu|$. For (iii) we have from (1.31) that

$$\begin{aligned}|\mu| = (\mu^+ - \mu^-) &= (\mu \vee 0) - (\mu \wedge 0)\\ &= \{\mu - (\mu \wedge 0)\} \vee (-(\mu \wedge 0)), && \text{by (1.31)}\\ &= (0 \vee \mu) \vee \{(-\mu) \vee 0\}, && \text{by (1.31)}\\ &= \{(\mu) \vee (-\mu)\}.\end{aligned}$$

Next from (iii) we have for any set $X \in B$ and for every partition $(A, \overline{X - A})$ of X

$$\begin{aligned}|\mu_1|(X) + |\mu_2|(X) &\geqslant \{\mu_1(A) - \mu_1(X - A)\} + \{\mu_2(A) - \mu_2(X - A)\}\\ &= (\mu_1 + \mu_2)(A) - (\mu_1 + \mu_2)(X - A),\end{aligned}$$

i.e. $\{|\mu_1| + |\mu_2|\}(X) \geqslant |\mu_1 + \mu_2|(X)$; hence $|\mu_1| + |\mu_2| \geqslant |\mu_1 + \mu_2|$.

Finally to prove (v), we have from definition

$$\begin{aligned}|\mu_1 - \mu_2| &= (\mu_1 - \mu_2)^+ - (\mu_1 - \mu_2)^-\\ &= \{[(\mu_1 - \mu_2) \vee 0] + \mu_2\} - \{[(\mu_1 - \mu_2) \wedge 0] + \mu_2\}\\ &= (\mu_1 \vee \mu_2) - (\mu_1 \wedge \mu_2), && \text{by (1.31).}\end{aligned}$$

DEFINITION 2. *Two measures μ, ν are said to be mutually singular, in symbols $\mu \perp \nu$, if* $(|\mu| \wedge |\nu|) = 0$.

With this definition, we shall now derive the usual definition of singularity of two measures μ, ν as a necessary and sufficient condition.

THEOREM 5. *A necessary and sufficient condition for $\mu \perp \nu$ is that, there exist a partition $(A, \overline{\Omega - A})$ of Ω such that for every set $E \in B$, $\mu(E \cap A) = 0$, $\nu(E \cap (\Omega - A)) = 0$.*

PROOF. By Theorem 3, if $(|\mu| \wedge |\nu|) = 0$, then we can find a partition $(A, \overline{\Omega - A})$ of Ω such that

$$|\mu|(A) + |\nu|(\Omega - A) = 0,$$

or $|\mu|(E \cap A) = 0$, $|\nu|(E \cap \overline{\Omega - A}) = 0$ for every set $E \in B$, which implies $\mu(E \cap A) = 0$, $\nu(E \cap \overline{\Omega - A}) = 0$ for every $E \in$ B. Hence the condition is necessary.

To prove that the condition is sufficient, let $(A, \overline{\Omega - A})$ be a partition of Ω satisfying the condition of the theorem. Then for every set $X \in B$

$$\begin{aligned}\{|\mu| \wedge |\nu|\}(X) &= \{|\mu| \wedge |\nu|\}(X \cap A) + \{|\mu| \wedge |\nu|\}(X \cap \overline{\Omega - A})\\ &= 0,\end{aligned}$$

since by hypothesis $|\mu|(X \cap A) = 0, |\nu|(X \cap \overline{\Omega - A}) = 0$. Thus $(|\mu| \Lambda |\nu|) = 0$.

This theorem means that if $\mu \perp \nu$, then the join[(4)] of the σ-ideals $I_{|\mu|}$ and $I_{|\nu|}$ is the entire family B.

THEOREM 6. *The positive and negative parts of any measure are mutually singular.*

For,

$$\begin{aligned} 0 = (-\mu^-) + (\mu \Lambda 0) &= \{(-\mu^- + \mu) \Lambda (-\mu^-)\}, && \text{by (1.31)} \\ &= \{\mu^+ \Lambda (-\mu^-)\}, && \text{by (1.32) with } \nu = 0 \\ &= \{|\mu^+| \Lambda |\mu^-|\}. \end{aligned}$$

THEOREM 7. *If* $\mu_1 \perp \nu$, $\mu_2 \perp \nu$, *then* $(\mu_1 + \mu_2) \perp \nu$.

PROOF. Let $(A, \overline{X - A})$, $(B, \overline{X - B})$ be the minimal partitions of $X \in B$ for $|\mu_1| \Lambda |\nu|$, $|\mu_2| \Lambda |\nu|$ respectively ($|\mu_1|(A) = 0$, $|\nu|(X - A) = 0$; $|\mu_2|(B) = 0$, $|\nu|(X - B) = 0$). Now if $C = A \cap B$, then the partition $(C, \overline{X - C})$ is such that $(\mu_1 + \mu_2)(S) = 0$ for every set $S \subset C$, while $\nu(S) = 0$ for every $S \subset \overline{X - C}$. By Theorem 5, it follows that $(\mu_1 + \mu_2) \perp \nu$.*

1.4. JORDON AND HAHN DECOMPOSITIONS

Setting $\nu = 0$ in (1.32) we deduce that, for any signed measure μ, $\mu = (\mu^+ + \mu^-)$. This representation of any signed measure μ as the sum of its positive and negative parts is called the *Jordon decomposition* of μ.

By virtue of singularity of μ^+ and μ^- (Theorem 5), we have a minimal partition $(A, \overline{\Omega - A})$ of Ω such that for every set $E \in B$, $\mu^+(E \cap \overline{\Omega - A}) = 0$ and $\mu^-(E \cap A) = 0$. It follows immediately that $\mu(E \cap A) \geqslant 0$ and $\mu(E \cap \overline{\Omega - A}) < 0$. This partition $(A, \overline{\Omega - A})$ gives the so-called *Hahn decomposition* of Ω with respect to the signed measure μ.

(4) The join of two σ-ideals J_1, J_2 is the family J consisting of all the elements $(X \cup Y)$, $X \in J_1$, $Y \in J_2$.

* The same partition $(C, \Omega - C)$ of Ω proves that if $\mu_1 \perp \nu$, $\mu_2 \perp \nu$, then $(\mu_1 \vee \mu_2) \perp \nu$.

REMARK 1. By corollary of Theorem 3, the minimal partition $(A, \overline{\Omega - A})$ of Ω for $(\mu \wedge \nu)$ has the property that $\mu(S) \leqslant \nu(S)$ for every set $S \subset A$ and $\nu(T) \leqslant \mu(T)$ for every set $T \subset (\overline{\Omega - A})$. It is thus clear that $(A , \overline{\Omega - A})$ is a Hahn decomposition for the measure $(\mu - \nu)$.

DEFINITION 3 : *The truncation of a measure μ over a set E, denoted by $\{\mu\}_E$, is a measure which is zero for every subset of $(\Omega - E)$, and is equal to μ for every subset of E.*

REMARK 2. If $(A , \overline{\Omega - A})$ be the maximal partition of Ω for $(\mu \vee \nu)$, then $(\mu \vee \nu)$ can be expressed as the sum of truncations of μ over the set A and of ν over the set $(\overline{\Omega - A})$;

$$(\mu \vee \nu) = \{\mu\}_A + \{\nu\}_{(\Omega - A)}.$$

1.5. INFINITE DISTRIBUTIVITY OF THE BOUNDEDLY COMPLETE LATTICE (μ)

We shall give, following Birkhoff page 231, a concise proof of infinite distributivity of (μ), by means of the group operation.

Let $\{\mu_\alpha\}$, where $\alpha \in \omega$ an abstract index set, represent a bounded family of measures.

Let $v = \bigvee_{\alpha \in \omega} \mu_\alpha$. Write $(v - \mu_\alpha) = t_\alpha \geqslant 0$.

Then for any $\mu_0 \in (\mu)$ and $\alpha \in \omega$ we have

$$\begin{aligned} 0 &\leqslant \{(\mu_0 \wedge v) - (\mu_0 \wedge \mu_\alpha)\} \\ &= \{t_\alpha + [(\mu_0 - t_\alpha) \wedge \mu_\alpha]\} - (\mu_0 \wedge \mu_\alpha) \leqslant t_\alpha, \text{ by } (1.31) \end{aligned}$$

or

$$0 \leqslant \bigwedge_{\alpha \in \omega} \{(\mu_0 \wedge v) - (\mu_0 \wedge \mu_\alpha)\} \leqslant \bigwedge_{\alpha \in \omega} (t_\alpha).$$

But

$$\bigwedge_{\alpha \in \omega} t_\alpha = v - \bigvee_{\alpha \in \omega} (\mu_\alpha) = 0, \qquad \text{by } (1.31).$$

Therefore

$$0 = \bigwedge_{\alpha \in \omega} \{(\mu_0 \wedge v) - (\mu_0 \wedge \mu_\alpha)\} = (\mu_0 \wedge v) - \bigvee_{\alpha \in \omega} (\mu_0 \wedge \mu_\alpha) \text{ by } (1.31)$$

Transposing we get

$$\mu_0 \wedge \{\bigvee_{\alpha \in \omega} \mu_\alpha\} = \bigvee_{\alpha \in \omega} (\mu_0 \wedge \mu_\alpha). \qquad (1.51)$$

The dual distributive law proved similarly is

$$\mu_0 \vee \{\bigwedge_{\alpha \in \omega} \mu_\alpha\} = \bigwedge_{\alpha \in \omega} (\mu_0 \vee \mu_\alpha). \tag{1.52}$$

Summarizing we have

THEOREM 8. *The infinite distributive laws* (1.51) *and* (1.52) *are valid for the system* (μ).

Finite distributivity of the lattice (μ) is a particular case of the above.

1.6. RELATIVE PRODUCT COMPLEMENTS. The lattice (μ) has no largest or smallest element. Therefore the sum and product complements of any element do not exist. However, since the system (μ) is a boundedly complete lattice, the complements of any element μ relative to a lower bound and an upper bound exist.[5]

THEOREM 9. *If* μ, $\nu > 0$, *then there exists a largest measure* ν_1, $(0 \leqslant \nu_1 \leqslant \nu)$ *such that* $\nu_1 \perp \mu$.

PROOF. There exist positive measures $\leqslant \nu$ singular with respect to μ; example 0. Consider the lattice sum ν_1 of all measures μ_α, $0 \leqslant \mu_\alpha \leqslant \nu$ such that $(\mu \wedge \mu_\alpha) = 0$. Then

$$(\mu \wedge \nu_1) = \mu \wedge \{\bigvee_\alpha \mu_\alpha\} = \bigvee_\alpha (\mu \wedge \mu_\alpha), \qquad \text{by (1.51).}$$

Hence $\mu \wedge \nu_1 = 0$, i.e. $\mu \perp \nu_1$.

DEFINITION 4. *A measure* ν *is said to be absolutely continuous with respect to* μ, *in symbols* $\nu \ll \mu$, *if the only positive measure* $\leqslant |\nu|$ *singular with respect to* μ *is* 0.

We may show that this definition of absolute continuity of ν with respect to μ is equivalent to the two usual definitions of absolute continuity, namely,

(i) for every set $E \in B$ for which $|\mu|(E) = 0$, $|\nu|(E) = 0$,

(5) If there is a largest element 1 and a smallest element 0 in a lattice L, then the sum and product complements of any element $x \in L$ are respectively element y_1, y_2 such that $y_1 \vee x = 1$ and $y_2 \wedge x = 0$, and $x \vee \mu = 1$, $x \wedge \nu = 0$ imply $\mu > y_1$ and $\nu < y_2$.

(ii) If μ, ν are signed measures, then $\nu \ll \mu$ if, corresponding to every positive number ϵ, there is a positive number δ such that $|\nu|(E) < \epsilon$ for every set $E \in B$ for which $|\mu|(E) < \delta$.

First we show that our definition is equivalent to (i).

Let $\nu \ll \mu$. Then $|\nu|(E) = 0$ whenever $|\mu|(E) = 0$. For, if for a set $E \in B$, $|\mu|(E) = 0$ but $|\nu|(E) > 0$, then the truncation of $|\nu|$ over $E\{|\nu|\}_E$, $0 \leqslant \{|\nu|\}_E \leqslant |\nu|$ is a measure singular with respect to μ (the partition of singularity is $(E, \overline{\Omega - E})$, $|\mu|(E) = 0$, $\{|\nu|\}_E(\Omega - E) = 0$). This contradicts the hypothesis that the only + ve measure $\leqslant |\nu| \perp$ w.r.t. μ is 0.

Conversely, let $|\nu|(E) = 0$ whenever $|\mu|(E) = 0$. Suppose that k, $(0 \leqslant k \leqslant |\nu|)$ is a measure $\perp \mu$, i.e. there exists a partition $(A, \overline{\Omega - A})$ of Ω such that $k(\Omega - A) = 0$, $|\mu|(A) = 0$. But $0 \leqslant k(A) \leqslant |\nu|(A) = 0$ by hypothesis. Therefore $k = 0$.

Next we show that (i) is equivalent to (ii). Let (i) be true. Suppose that it is possible, for some $\epsilon > 0$, to find a sequence $\{E_i\}$ of sets $\in B$ such that $|\mu|(E_n) < 1/2^n$ and $|\nu|(E_n) \geqslant \epsilon$, $n = 1, 2, \ldots$. If $E = \limsup_n E_n$, then $|\mu|(E) \leqslant \sum_{i=n}^{\infty} |\mu|(E_i) < 1/2^{n-1}$, $n = 1, 2, \ldots$ and therefore $|\mu|(E) = 0$. On the other hand

$$|\nu|(E) = \lim_n |\nu|\{\bigcup_{i=n}^{\infty} E_i\} \geqslant \limsup_n \{|\nu|(E_n)\} \geqslant \epsilon.$$

Since this contradicts the relation (i), (i) implies (ii).

The other part is easily proved. For, if for a set $E \in B$, $|\mu|(E) = 0$ but $|\nu|(E) = t > 0$, then (ii) is violated for $\epsilon = t$. Hence the proof is complete.

By Theorem 9, if $\mu, \nu \geqslant 0$, the largest measure ν_1 $(0 \leqslant \nu_1 \leqslant \nu)$, singular with respect to μ exists. Now, consider the decomposition of ν defined by

$$\nu = \nu_1 + (\nu - \nu_1). \tag{1.53}$$

We now assert that $(\nu - \nu_1) \ll \mu$; i.e. the only possible positive measure $\leqslant (\nu - \nu_1)$ which is singular with respect to μ is 0. For, if k $(0 \leqslant k \leqslant \nu - \nu_1)$ is a measure different from 0 such that

$k \perp \mu$, then by Theorem 7, $(k + \nu_1)$ is also $\perp \mu$. But $\nu_1 \leqslant (k + \nu_1) < \nu$, which contradicts the assumption that ν_1 is the largest measure $< \nu$, with $\nu_1 \perp \mu$.

This decomposition of ν into a singular and absolutely continuous part, with respect to another measure μ is called the *Lebesgue decomposition.* The Lebesgue decomposition of ν with regard to μ, when μ, ν are signed measures, may be obtained by adding the decompositions of ν^+, ν^- with respect to $|\mu|$.

We give in corollary of Theorem 12, another characterization, by limits, of absolute continuity which will be useful for our discussion of the Radon-Nikodym theorem.

II. The system (μ) as a Banach-lattice.

2.1. It has been shown in (1.3) that the system (μ) is a vector lattice. In this section we go further and show that (μ) is a Banach-lattice.

DEFINITION 5. *A vector lattice (μ) which is a Banach-space with norm $\|\mu\|$ and satisfies the condition*

$$|\mu| \leqslant |\nu| \text{ implies } \|\mu\| \leqslant \|\nu\|, \tag{2.11}$$

is called a Banach-lattice.

THEOREM 10. *If we define a norm in (μ) as $\|\mu\| = |\mu|(\Omega)$, (μ) becomes a Banach-lattice in which, if $\mu \geqslant 0, \nu \geqslant 0$, then*

$$\|\mu + \nu\| = \|\mu\| + \|\nu\|. \tag{2.12}$$

PROOF. From (i), (ii) and (iv) of Theorem 4, we have immediately

$$\|\mu\| > 0, \quad \|\mu\| = 0 \quad \text{if and only if} \quad \mu \doteq 0;$$

$$\|c\mu\| = |c|\,\|\mu\|; \quad \|\mu + \nu\| \leqslant \|\mu\| + \|\nu\|.$$

Next we show the completeness of (μ) with the norm metric. For any set $C \in B$ we have

$$0 \leqslant |(\mu_n - \mu_m)(C)| \leqslant |\mu_n - \mu_m|(C) \leqslant |\mu_n - \mu_m|(\Omega) = \|\mu_n - \mu_m\|. \tag{2.13}$$

We shall use this inequality in proving the completeness of the metric space (μ). If $\{\mu_n\}$ is a norm-fundamental sequence from (μ), then it is clear from (2.13) that $|(\mu_n - \mu_m)C| \to 0$ uniformly for all

sets $C \in B$; hence the uniform limit of $\{\mu_n(C)\}$ may be defined as μ (C), for every set $C \in B$.

To prove that μ is a measure, consider a partition $\{ X_i ; i = 1, 2, \ldots \}$ of $X \in B$. We have, for any integers n, k

$$|\mu(X) - \mu(\bigcup_{i=1}^{k} X_i)|$$
$$< |\mu(X) - \mu_n(X)| + |\mu_n(X) - \mu_n(\bigcup_{i=1}^{k} X_i)| + |\mu_n(\bigcup_{i=1}^{k} X_i) - \mu(\bigcup_{i=1}^{k} X_i)| \quad (2.14)$$

Now given $\epsilon > 0$, we can choose $n_0 = n_0(\epsilon)$ such that the first and third terms on the right side of (2.14) are $< \frac{\epsilon}{3}$. Now choose $k_0 = k_0(n_0, \epsilon)$ so large that the second term $< \frac{\epsilon}{3}$ (this is possible since μ_n is continuous).

Thus

$$|\mu(X) - \mu(\bigcup_{i=1}^{k} X_i)| < \epsilon, \text{ for all } k > k_0. \quad (2.15)$$

Hence

$$\mu(X) = \sum_{i=1}^{\infty} \mu_i(X_i)$$

Again, for any integer n we have

$$\| \mu_n - \mu \| = |\mu_n - \mu|(\Omega) = (\mu_n - \mu)(A_n) + (\mu - \mu_n)(\Omega - A_n), \quad (2.16)$$

$(A_n, \overline{\Omega - A_n})$ being the maximal partition of Ω for

$$|\mu_n - \mu| = (\mu_n - \mu) \vee (\mu - \mu_n).$$

Now given $\epsilon > 0$, we can choose n_0 such that for all $n > n_0$

$$|(\mu_n - \mu)(A_n)| < \epsilon/2, \quad |(\mu_n - \mu)(\Omega - A_n)| < \epsilon/2.$$

Hence it follows from (2.16) that

$$|\mu_n - \mu|(\Omega) < \epsilon \text{ for all } n > n_0; \text{ i.e } \| \mu_n - \mu \| \to 0 \text{ as } n \to \infty. \quad (2.17)$$

Hence (μ) is complete.

Finally, if $\mu > 0$, then clearly $\|\mu\| = \mu(\Omega)$. This immediately implies (2.12).

To prove (2.11), we note that, if $|\mu| \leqslant |\nu|$ we get from (2.12)

$$\||\nu|\| = \||\nu| - |\mu|\| + \||\mu|\| \geqslant \||\mu|\|;$$

but the definition of norm implies $\||\mu|\| = \|\mu\|$. Hence the result.

2.2. Species of convergence. We may list the following types of convergence along with norm convergence in the system of measures (μ).

(*a*) Monotonic convergence :

A sequence $\{\mu_n\}$ of measures is said to converge montonically to a measure μ, if $\mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \dots (\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \dots)$ and $\mu = \bigvee_{i=1}^{\infty} \mu_i$ (and $\mu = \bigwedge_{i=1}^{\infty} \mu_i$).

(*b*) Order-convergence :

If $\{\mu_n\}$ is a bounded sequence of measures from the boundedly σ-complete lattice (μ), it has a $\limsup_n (\mu_n)$ and $\liminf_n (\mu_n)$ defined by

$$\limsup_n \{\mu_n\} = \bigwedge_n \bigvee_k (\mu_{n+k})$$
$$\liminf_n \{\mu_n\} = \bigvee_n \bigwedge_k (\mu_{n+k})$$

The sequence $\{\mu_n\}$ is said to order-converge to a measure μ if $\mu = \limsup_n (\mu_n) = \liminf_n (\mu_n)$.

(*c*) Set-wise convergence:

A sequence of measures $\{\mu_n\}$ is said to converge set-wise to a measure μ if $\mu_n(C) \to \mu(C)$ for every set $C \in B$.

(*d*) Uniform set-wise convergence:

A sequence of measures $\{\mu_n\}$ is said to converge set-wise uniformly to a measure μ, if $\mu_n(C) \to \mu(C)$ uniformly for all sets $C \in B$; i.e. given $\epsilon > 0$, we can find an integer n_0 such that, whatever be the set $X \in B$

$$|\mu_n(X) - \mu(X)| < \epsilon \text{ for all } n > n_0.$$

(e) Norm convergence:

A sequence of measures $\{\mu_n\}$ is said to norm-converge to a measure μ if $\|\mu_n - \mu\| \to 0$ as $n \to \infty$. (Refer Theorem 10, for definition of norm).

THEOREM 11. (i) *Norm convergence is equivalent to uniform set-wise convergence.*

PROOF : From (2.13) we have, for any integer n,

$$0 \leqslant |(\mu_n - \mu)(C)| \leqslant |\mu_n - \mu|(C) \leqslant |\mu_n - \mu|(\Omega) = \|\mu_n - \mu\|,$$
for every set $C \in B$.

Therefore, if $\mu_n \xrightarrow{\text{norm}} \mu$, it is immediate from this, that $|(\mu_n - \mu)(C)| \underset{n\to\infty}{\to} 0$ uniformly for all sets $C \in B$.

To prove the converse, if $(A_n, \overline{\Omega - A_n})$ be the maximal partition of Ω for $|\mu_n - \mu| = (\mu_n - \mu) \vee (\mu - \mu_n)$, then

$$|\mu_n - \mu)|(\Omega) = (\mu_n - \mu)(A_n) + (\mu - \mu_n)(\Omega - A_n). \quad (2.21)$$

Now, given $\epsilon > 0$, since $\mu_n \to \mu$ set-wise uniformly, we can find an integer n_0 such that $|(\mu_n - \mu)(X)| < \epsilon/2$ for $n > n_0$, whatever be the set $X \in B$.

Thus we immediately have from (2.21)

$$|\mu_n - \mu|(\Omega) < \epsilon \text{ for } n > n_0,$$

i.e.

$$\|\mu_n - \mu\| \to 0 \text{ as } n \to \infty.$$

(ii) *Monotonic convergence implies norm-convergence.*

We shall prove this result for the case of monotonic decreasing sequence of measures (μ_n). The proof for monotonic increasing sequence (μ_n) is similar.

We may suppose without any loss of generality that μ_n's are positive and decrease to 0; for otherwise we may consider $(\mu_n - \bigwedge_{n=1}^{\infty} \mu_n)$ instead of (μ_n).

We now assert that $\mu_n(\Omega)$ decreases to 0, as $n \to \infty$. For, suppose it is possible that $\mu_n(\Omega) \to \epsilon > 0$, i.e. $\mu_n(\Omega) \geqslant \epsilon$ for all n.

Since by hypothesis $(\bigwedge_{n=1}^{\infty} \mu_n) = 0$, by definition of lattice-product we can find a countable partition $\{X_i\}$ of Ω such that

$$\sum_{i=1}^{\infty} \mu_i(X_i) < \epsilon/2. \tag{2.22}$$

Now, since each μ_n is continuous, there is an integer k' so that for $k \geqslant k'_{(n)}$

$$\mu_n(\bigcup_{i=1}^{k} X_i) \geqslant \mu_n(\Omega) - \epsilon/2 \geqslant \epsilon/2, \{k'_1 \leqslant k'_2 \leqslant \ldots\}. \tag{2.23}$$

But

$$\sum_{1}^{\infty} \mu_i(X_i) \geqslant \mu_n(\bigcup_{i=1}^{k'_1} X_i) \text{ for } n \geqslant k'_1,$$

$$\geqslant \epsilon/2, \qquad \text{from (2.23)},$$

which contradicts (2.22). Hence $\mu_n(\Omega) \to 0$ as $n \to \infty$; i.e. $\|\mu_n\| \to 0$ as $n \to \infty$. The proof is complete.

Corollary. *If μ, ν are positive measures, $(n\mu \wedge \nu)$ forms a monotonic increasing sequence of measures bounded above by ν.*

Therefore by (ii) $(n\mu \wedge \nu)$ norm-converges to $\bigvee_{i=1}^{\infty} (n\mu \wedge \nu) \leqslant \nu$. The identification of this limit is taken up in the next theorem.

We have shown in (1.6) that ν_1 (the largest measure, $0 \leqslant \nu_1 \leqslant \nu$, singular with respect to μ), and $(\nu - \nu_1)$ constitute the Lebesgue decomposition of ν with respect to μ. We now show that $(\nu - \nu_1)$ can be exhibited as the norm limit of the increasing sequence of measures $(n \mu \wedge \nu)$.

Theorem 12. *If μ, ν are positive measures and ν_1 is the singular part of ν with respect to μ (and $(\nu - \nu_1)$ the absolutely continuous part of ν with respect to μ), then $(n\mu \wedge \nu) \to$ in norm to $(\nu - \nu_1)$.*

Proof. We have already proved in Theorem 11 (i) that norm convergence is equivalent to uniform set-wise convergence; hence it suffices to show here that $\{n\mu \wedge \nu\} \to (\nu - \nu_1)$ set-wise uniformly.

Let $(C, \overline{\Omega - C})$ be the partition of Ω arising from the singularity of ν_1 and μ, where $\nu_1(C) = 0$ and $\mu(\Omega - C) = 0$; (this implies $\mu(C) > 0$

for otherwise $\mu = 0$). Let $(A_n, \overline{C - A_n})$ be a minimal partition of C for $\{n\mu \Lambda \nu\}$, $n = 1, 2, \ldots$. On account of the corollary of Theorem 3, we have for the partition $(A_n, C - A_n)$

$$\left.\begin{aligned} \nu(P) &\leqslant n\,\mu(P), \text{ for every } P \subset (C - A_n), \\ \nu(P) &\geqslant n\,\mu(P), \text{ for every } P \subset (A_n). \end{aligned}\right\} \quad (2.24)$$

We now show that the minimal partitions can be so adjusted that (A_n) forms a decreasing sequence of sets; for, if P were any subset of $(A_n \cap A^*_{n-1})$, then from (2.24) we have

$$n\,\mu\,(P) \leqslant \nu\,(P) \leqslant (n-1)\,\mu\,(P),$$

which implies

$$\mu\,(P) = \nu\,(P) = 0;$$

therefore $(A_n \cap A^*_{n-1})$ can be amalgamated with $(C - A_n)$ without affecting the minimal character of $(A_n, \overline{C - A_n})$.

Thus we may suppose without loss of generality that the sets (A_n) form a decreasing sequence. Let $A = \bigcap_{i=1}^{\infty} A_i$. We now assert that $\mu\,(A) = 0$. For, since $\nu \geqslant (n\,\mu\,\Lambda\,\nu)$ we have

$$\begin{aligned} \nu(C) &\geqslant n\,\mu(A_n) + \nu(C - A_n) \\ &\geqslant n\,\mu\,(A) + \nu\,(C - A_n) \geqslant n\,\mu\,(A); \end{aligned}$$

if $\mu\,(A)$ were > 0 we can choose n so large that this inequality is invalidated. Now, since $\mu(A) = 0$ and $(\nu - \nu_1) \ll \mu$, $\nu(A) = \nu_1(A)$; but $\nu_1(A) \leqslant \nu_1\,(C) = 0$ by hypothesis. Hence $\nu(A) = 0$.

Therefore

$$\nu(A_n) \geqslant \{\nu(A_n) - n\,\mu(A_n)\} = \{\nu - (n\,\mu\,\Lambda\,\nu)\}\,(C) \geqslant 0.$$

Taking the limit as $n \to \infty$ we have, on account of continuity of ν

$$\begin{aligned} 0 = \lim_{n\to\infty} \nu(A_n) &= \lim_{n\to\infty} \{\nu - (n\,\mu\,\Lambda\,\nu)\}\,(C) \\ &= \lim_{n\to\infty} \{\nu - \nu_1) - (n\,\mu\,\Lambda\,\nu)\}\,(C). \quad (2.25) \end{aligned}$$

* Denotes complementation

Next to prove that the convergence of $(n\mu \wedge \nu)$ to $(\nu - \nu_1)$ is uniform, let R be any set $\in B$. Then

$$\{(\nu - \nu_1) - (n\mu \wedge \nu)\}(R) = \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(R \cap C) + \\ + \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(R \cap \overline{\Omega - C}). \quad (2.26)$$

Now since $\mu(\Omega - C) = 0$ and $(\nu - \nu_1)$ and $(n\mu \wedge \nu)$ are both $\ll \mu$, the 2nd term on the right side of (2.26) vanishes giving

$$0 \leqslant \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(R) = \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(R \cap C).$$

Here again since $\nu_1(C) = 0$ we have

$$\begin{aligned} 0 \leqslant \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(R) &= \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(R \cap C) \\ &= \{\nu - (n\mu \wedge \nu)\}(R \cap C) \\ &\leqslant \{\nu - (n\mu \wedge \nu)\}(C) \\ &= \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(C). \quad (2.27) \end{aligned}$$

Hence from (2.25), given $\epsilon > 0$ we can find an integer n_0 so that for $n > n_0$,

$$0 \leqslant \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(R) \leqslant \{(\nu - \nu_1) - (n\mu \wedge \nu)\}(C) < \epsilon,$$

which implies that $(n\mu \wedge \nu) \to (\nu - \nu_1)$ uniformly for all sets $R \in B$.

As a particular case of this theorem we have

COROLLARY. *If $\mu, \nu \geqslant 0$, then $\nu \ll \mu$ if and only if $(n\mu \wedge \nu)$ converges in norm to ν.*

2.3. As in Theorem 12, if $\mu, \nu > 0$ and $\nu \ll \mu$, we can construct minimal partitions $(A_n, \overline{\Omega - A_n})$ of Ω for $(n\mu \wedge \nu)$, with the sequence of sets (A_n) decreasing. We shall utilize the partitions $(A_n, \overline{\Omega - A_n})$ in proving the following theorem.

THEOREM 13. *If $\mu, \nu \geqslant 0$ and $\nu \ll \mu$, then given a positive number ϵ however small, there exists a finite partition $(C_1, C_2, \ldots, C_{n_0})$ of Ω and a set of positive constants $(\lambda_1, \lambda_2, \ldots, \lambda_{n_0})$ such that for every set $T \in B$*

$$\left|\nu(T) - \sum_{i=1}^{n_0} \lambda_i \mu(C_i \cap T)\right| < \epsilon.$$

PROOF. We note in the first place that $\nu \ll \mu$ implies $\nu \ll \frac{\epsilon}{2\mu(\Omega)}\mu$.

Let $(A_i, \Omega - A_i)$ $i = 1, 2 \ldots$ be the minimal partitions of Ω for $\left(\frac{i\epsilon}{2\mu(\Omega)}\mu \Lambda \nu\right)$, so that for every subset T of A_i

$$\nu(T) \geqslant \frac{i\epsilon}{2\mu(\Omega)}\mu(T), \tag{2.31}$$

and for every subset T of $(\Omega - A_i)$, $\nu(T) \leqslant \frac{i\epsilon}{2\mu(\Omega)}\mu(T)$.

We may suppose as in Theorem 12 that (A_i) is a decreasing sequence. It follows as in the proof of Theorem 12 that $\mu(A) = 0$ (and therefore $\nu(A) = 0$) where $A = \bigcap_{i=1}^{\infty} A_i$. Now given a positive number ϵ, we choose n_0 such that

$$\nu(A_n) < \epsilon/2, \text{ for } n \geqslant (n_0 - 1). \tag{2.32}$$

Write

$$C_1 = (\Omega - A_1)$$
$$C_2 = (\Omega - A_2) - (\Omega - A_1)$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$C_{n_0-1} = (\Omega - A_{n_0-1}) - (\Omega - A_{n_0-2})$$
$$C_{n_0} = A_{n_0-1}.$$

From 2.31 we have the following relations between ν and μ for subsets of C_i:

$$\frac{(i-1)\epsilon}{2\mu(\Omega)}\mu(S) \leqslant \nu(S) \leqslant \frac{i\epsilon}{2\mu(\Omega)}\mu(S),\ i = 1, 2, \ldots (n_0 - 1),$$

and

$$\frac{(n_0 - 1)\epsilon}{2\mu(\Omega)}\mu(S) \leqslant \nu(S) \text{ for } S \subset C_{n_0}. \tag{2.33}$$

Define the coefficients λ_i as follows:

$$\lambda_i = \sup_{\{S\}} \frac{\nu(S)}{\mu(S)}, \text{ for all sets } S \subset C_i \text{ and } \mu(S) > 0$$
$$= 0, \text{ if } \mu(C_i) = 0,\ i = 1, 2, \ldots (n_0 - 1),$$

and $$\lambda_{n_0} = 0.$$

From (2.33) it is clear that $\frac{(i-1)\epsilon}{2\mu(\Omega)} \leqslant \lambda_i \leqslant \frac{i\epsilon}{2\mu(\Omega)}$ if $\mu(C_i) < 0$ and if $\mu(C_i) = 0$ (and therefore $\nu(C_i) = 0$), $\lambda_i = 0$, $i = 1, 2, \ldots (n_0 - 1)$. (2.34)

Thus since $\nu(\Omega - C_{n_0}) = \sum_{i=1}^{n_0-1} \left\{ \frac{\nu(C_i)}{\mu(C_i)} \right\} \mu(C_i)$ (without affecting the proof we may assume that all $\mu(C_i) > 0$)

$$\left| \nu(\Omega - C_{n_0}) - \sum_1^{n_0-1} \lambda_i \, \mu(C_i) \right| < \frac{\epsilon \, \mu(\Omega - C_{n_0})}{2\mu(\Omega)}$$

$$< \frac{\epsilon}{2},$$

from (2.33) and (2.34).

Hence

$$\left| \nu(\Omega) - \sum_1^{n_0} \lambda_i \mu(C_i) \right| \leqslant \left| \nu(\Omega - C_{n_0}) - \sum_1^{n_0-1} \lambda_i \mu(C_i) \right| + \left| \nu(C_{n_0}) \right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \qquad \text{by (2.32).}$$

We have thus shown the existence of a partition $(C_1, C_2, \ldots, C_{n_0})$ of Ω and a definite function $\sum_1^{n_0} \lambda_i \, \mu(C_i)$ differing from $\nu(\Omega)$ by $< \epsilon$, With this partition $(C_1, C_2 \ldots, C_{n_0})$ and the same set of coefficients $(\lambda_1, \lambda_2, \ldots, \lambda_{n_0})$ we have for any set $T \in B$

$$\left| \nu(T) - \sum_1^{n_0} \lambda_i \, \mu(C_i \cap T) \right| \leqslant \left| \sum_1^{n_0-1} \left\{ \frac{\nu(C_i \cap T)}{\mu(C_i \cap T)} - \lambda_i \right\} \times \right.$$

$$\left. \times \, \mu(C_i \cap T) \right| + \left| \nu(T \cap C_{n_0}) \right|$$

$$< \frac{\epsilon}{2\,\mu(\Omega)} \, \mu(T - \overline{T \cap C_{n_0}}) + \frac{\epsilon}{2}$$

$$< \epsilon, \qquad \text{by (2.32) and (2.33).}$$

COROLLARY. *Similar to the partition $P_1 = (C_1, C_2, \ldots, C_{n_0})$ of Ω constructed above, we have corresponding to $\epsilon/2$ another partition $P_1 = (D_1, D_2, \ldots, D_m)$ $(m > n_0)$ of Ω where m is so chosen that $\nu(D_m) < \epsilon/2^2$. We now show that the partition $(P_1, P_2) = (C_i \cap D_j;$ $i = 1, 2, \ldots, n_0, j = 1, 2, \ldots, m)$ will serve for $\epsilon/2$ instead of P_2.*

For, define the coefficients α_{ij} corresponding to the sets $(C_i \cap D_j)$ as follows :

$$\alpha_{ij} = \sup_{\{S\}} \frac{\nu(S)}{\mu(S)}, \qquad S \subset (C_i \cap D_j) \text{ and } \mu(s) > 0,$$

$$= 0, \quad \text{if} \quad \mu(C_i \cap D_j) = 0, \quad i = 1, 2, \ldots, n_0,$$

$$j = 1, 2, \ldots, (m-1),$$

and

$$\alpha_{im} = 0, \qquad \text{for all } i. \tag{2.35}$$

Similar to the relations (2.33) for the partition P_1 we would have the following relations between μ and ν for subsets of D_i:

$$\frac{(i-1)\,\epsilon}{2^2\mu(\Omega)}\,\mu(S) \leqslant \nu(S) \leqslant \frac{i\,\epsilon}{2^2\mu(\Omega)}\,\mu(S),\; i = 1, 2, \ldots, (m-1) \tag{2.36}$$

and

$$\frac{(m-1)\,\epsilon}{2^2\,\mu(\Omega)}\,\mu(S) \leqslant \nu(S) \text{ for } S \subset D_m;$$

where m is chosen so that $\nu(D_m) < \epsilon/2^2$. Hence we have for any set $T \in B$

$$\Big|\nu(T) - \sum_{i=1}^{n_0} \sum_{j=1}^{m} \alpha_{ij}\,\mu(C_i \cap D_j \cap T)\Big|$$

$$\leqslant \Big|\nu\{T - \overline{T \cap D_m}\} - \sum_{i=1}^{n} \sum_{j=1}^{m-1} \alpha_{ij}\,\mu(C_i \cap T)\Big| +$$

$$+ \big|\nu(T \cap D_m)\big|, \text{since } \alpha_{im} = 0 \text{ by (2.35)},$$

$$= \Big|\sum_{i=1}^{n_0} \sum_{j=1}^{m-1} \Big\{\frac{\nu(T \cap C_i \cap D_j)}{\mu(T \cap C_i \cap D_j)} - \alpha_{ij}\Big\}\,\mu(C_i \cap D_j \cap T)\Big| +$$

$$+ \big|\nu(T \cap D_m)\big|. \tag{2.37}$$

Now by virtue of (2.36) we have $\left|\dfrac{\nu(T \cap C_i \cap D_j)}{\mu(T \cap C_i \cap D_j)} - \alpha_{ij}\right| < \dfrac{\epsilon}{2^2}\,\dfrac{1}{\mu(\Omega)}$.

Using this and the fact that $\nu(D_m) < \epsilon/2^2$ in (2.37) we have

$$\Big|\nu(T) - \sum_{i=1}^{n_0} \sum_{j=1}^{m} \alpha_{ij}\,\mu(C_i \cap D_j \cap T)\Big| < \epsilon/2.$$

Thus we can choose a refinement (P_1, P_2) of P_1, to correspond to $\epsilon/2$ and this process can be continued indefinitely. Hence $\nu(T)$, for

any set $T \in B$, can be approached by means of successive refinements of the original partition.

We may state Theorem 13 in a slightly different form as follows:

THEOREM 14. *Given a positive number ϵ however small, there exists a definite linear combination of truncations of μ such that for every set $T \in B$*

$$\left|\{\nu - [\sum_{1}^{n_0} \lambda_i \{\mu\}_{C_i}]\} (T)\right| < \epsilon.$$

III. The Radon-Nikodym theorem

3.1. DEFINITION 6. *If $C \in B$ be any set, the function χ_C defined for all points x in Ω by the relations*

$$\chi_C(x) = \begin{cases} 1, & \text{if } x \in C, \\ 0, & \text{if } x \in (\Omega - C), \end{cases}$$

is called the characteristic function of the set C.

Any finite linear combination of characteristic functions is called a *simple function.* It is easy to see that any simple function can be expressed as a linear combination of characteristic functions of disjoint sets. The integral of a simple function $f(x) = \sum_{i=1}^{n} \alpha_i \chi_{C_i}(x)$, with respect to the bounded signed measure μ over any set $G \in B$, is defined as

$$\int_G \{\Sigma\, \alpha_i\, \chi_{C_i}(x)\}\, d\mu = \sum_{i=1}^{n} \alpha_i\, \mu(C_i \cap G).$$

A finite valued measurable (B) function $f(x)$ on Ω is integrable with respect to a bounded measure μ, if there exists a mean fundamental sequence of simple functions $(f_i(x))$ (i.e. a sequence of simple functions $(f_i(x))$ such that $\int_\Omega |f_n - f_m|\, d\mu \to 0$ as $n, m \to \infty$) which converges in measure (μ) to $f(x)$. The integral of $f(x)$, $\int f(x) d\mu$ is defined by $\int f(x)\, d\mu = \lim_{n\to\infty} \int f_n(x)\, d\mu$ the latter limit existing in virtue of our hypothesis.

Theorem 15. *Let $\mu, \nu \geqslant 0$ and $\nu \ll \mu$. Then given a positive number ϵ however small, there exists a definite simple function $f(x)$ such that for every set $T \in B$, $|\nu(T) - \int_T f(x)\, d\mu| < \epsilon$.*

Proof. The proof is immediate from Theorem 13, since

$$\int_T \{\sum_{i=1}^{n_0} \lambda_i \chi_{C_i}(x)\}\, d\mu = \sum_{i=1}^{n_0} \lambda_i\, \mu(C_i \cap T).$$

Lemma. *Let (P_i), $i = 1, 2, \ldots$, $P_i = (C_{1i}, C_{2i}, \ldots, C_{n_i i})$, $(n_1 < n_2 < \ldots)$ be a sequence of partitions of Ω each being a refinement of the preceding, such that for every set $T \in B$, $|\nu(T) - \int_T f_i(x)\, d\mu| < \epsilon/2^i$ where $f_i(x)$ is the simple function corresponding to the partition P_i (the existence of such a sequence of partitions (P_i) and of the corresponding simple functions (f_i) has been established in Theorem* 13 *and corollary, and Theorem* 15). *Then*

(i) *the sequence of simple functions $\{f_i(x)\}$ converges everywhere to a definite measurable (B) function $f(x)$;*

(ii) *The sequence $\{f_i(x)\}$ is mean fundamental, i.e.*

$$\int |f_m - f_n|\, d\mu \to 0 \text{ as } n, m \to \infty.$$

The question arises as to the degree of determinateness of the function $f(x)$. Since the partitions in question are indeterminate to the extent of sets of μ-measure zero which can be shuffled from one part to the other without affecting their essential character, it follows that any two such functions $f(x)$ will agree except on a set of μ-measure zero.

Proof. Let $C_{11}, C_{12}, C_{13}, \ldots$, be the sequence of sets $(C_{1i} \in P_i)$ each containing a given point $x \in \Omega$. Since P_i is a refinement of P_{i-1} it follows that $C_{11} \supset C_{12} \supset C_{13} \supset \ldots$. Then the coefficients of the simple functions $\{f_i(x)\}$ corresponding to these sets, namely,

$$\begin{cases} \sup_S \dfrac{\nu(S)}{\mu(S)} & \text{for } S \subset C_{1i}, \mu(S) > 0, \\ 0, & \text{if} \quad \mu(C_{1i}) = 0, \end{cases}$$

necessarily form a decreasing sequence (at least after a certain stage) of real numbers tending to a definite limit $\geqslant 0$. Define $f(x)$ at each $x \in \Omega$ as equal to this limit. Thus $f(x) = \lim_{n \to \infty} f_n(x)$ at each $x \in \Omega$. Hence $f(x)$ is the everywhere limit of $\{f_n(x)\}$ (which implies that $f_i(x) \to f(x)$ in μ-measure). It follows immediately that $f(x)$ is measurable (B).

Next from Theorem 15 it is clear that the sequence $\{\int_T f_i(x)\, d\mu\}$ is a uniformly fundamental sequence, i.e. given $\eta > 0$, there exist integers n_0, m_0 such that for every set $T \in B$

$$\left| \int \{f_n(x) - f_m(x)\}\, d\mu \right| < \eta, \quad \text{for } n \geqslant n_0,\ m \geqslant m_0. \qquad (3.11)$$

To prove that $(f_i(x))$ is mean fundamental, let

$$(f_n(x) - f_m(x)) \geqslant 0, \quad \text{if } x \in C_1 \subset T,$$
$$(f_n(x) - f_m(x)) \leqslant 0, \quad \text{if } x \in (T - C_1).$$

Applying (3.11) to each of the sets C_1, $(T - C_1)$, we have for $n \geqslant n_0$, $m \geqslant m_0$,

$$\left| \int_{C_1} (f_n(x) - f_m(x))\, d\mu \right| = \int_{C_1} |f_n(x) - f_m(x)|\, d\mu < \eta, \qquad (3.12)$$

and

$$\left| \int_{(T-C_1)} (f_n(x) - f_m(x))\, d\mu \right| = \int_{(T-C_1)} |f_n(x) - f_m(x)|\, d\mu < \eta. \qquad (3.13)$$

Combining (3.12) and (3.13) we get

$$\int_T |f_n - f_m|\, d\mu = \int_{C_1} |f_n - f_m|\, d\mu + \int_{(T-C_1)} |f_n - f_m|\, d\mu$$
$$< 2\eta, \quad \text{for } n \geqslant n_0,\ m \geqslant m_0.$$

Hence

$$\int_T |f_n - f_m|\, d\mu \to 0 \text{ as } n, m \to \infty,$$

uniformly for all sets $T \in B$. In particular (f_n) is a mean fundamental sequence. Since (f_n) converges everywhere to $f(x)$, it is clear that $f(x)$ is integrable (μ). Thus

$$\int_T f_n(x)\, d\mu \to \int_T f(x)\, d\mu, \quad \text{uniformly for all sets } T \in B.$$

But

$$\int_T f_n(x)\, d\mu \to \nu(T).$$

Therefore

$$\int_T f(x)\, d\mu = \nu(T),$$

for every set $T \in B$. Summarizing we have,

THEOREM 16. *If μ, $\nu \geqslant 0$ and $\nu \ll \mu$, then there exists an integrable function $f(x)$ such that for every set $T \in B$*

$$\nu(T) = \int_T f(x)\, d\mu. \tag{3.14}$$

This is the celebrated theorem of Radon-Nikodym.

The theorem holds if $\mu > 0$ but ν is a signed measure. For, $\nu \ll \mu$ is equivalent to the simultaneous validity of the conditions

$$\nu^+ \ll \mu, \quad \nu^- \ll \mu$$

and

$$\begin{aligned}\nu(T) &= \nu^+(T) + \nu^-(T)\\ &= \int_T f_1(x)\, d\mu + \int_T f_2(x)\, d\mu,\end{aligned}$$

applying Theorem 16 to each part,

$$= \int_T \{f_1(x) + f_2(x)\}\, d\mu.$$

The Radon-Nikodym theorem remains true even if μ is only a signed measure. For let $T = (A_1 \cup A_2)$ be the Hahn-decomposition with respect to μ. Applying the theorem separately to ν and μ^+ in A_1 and to ν and μ^- in A_2, we get the Radon-Nikodym theorem for the general case.

It is clear from the lemma above that the Radon-Nikodym derivative $f(x)$ of ν w.r.t. μ defined as the limit of the coefficients of the simple functions $\{f_i(x)\}$ for each point x, will be bounded if there exists an integer n such that $n\mu \geqslant \nu$; otherwise the Radon-Nikodym derivative $f(x)$ will be unbounded.

REPORT OF THE TWENTIETH CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY

THE INDIAN MATHEMATICAL SOCIETY

TWENTIETH CONFERENCE, HYDERABAD 1954

LEFT TO RIGHT

Sitting on the ground : S. Pankajam, K. N. Kamalamma, Srirangamma, Nagarathnam, N. Padma, Marakata Krishnan.

Sitting on Chair : M. Sayeeduddin, S. Bhagavantam, Wade Ellis, H. Maass, G. S. Melkote, Gopal Rao Ekbote, Ram Behari, S. V. Ramamurthy, S. Mahadevan, Mrs. Rademacher, H. Rademacher, A. Narasinga Rao, D. D. Kapadia, V. P. Venkatachari.

Standing 1st Row : B. S. Madhava Rao, M. S. Narasimhan, S. Meenakshisundaram, S. S. Subramanyam, K. Subba Rao, N. Rama Rao, D. K. Sen, S. Swaminathan, M. R. Parameswaran, M. S. Ramanujan, M. Venkataraman, K. M. Shah, G. C. Patni, N. Manandhar, V. Ganapathy Iyer, K. R. Gunjikar, C. N. Srinivasiengar, P. L. Bhatnagar, R. P. Bambah, V. S. Krishnan.

Standing 2nd Row : K. Balagangadharan, A. R. Kamat, T. V. Avadhani, Shamsher Bahadur, A. R. Rao, N. T. Seth, C. C. Shah, L. N. Subramanyam, C. Venkoba Rao, N. Sethuraman, K. Seetharaman, R. Shankar, V. S. Huzurbazzar, R. S. Misra, O. B. Shukla, K. M. Saksena, R. N. Kaul, M. K. Kulkarni.

Standing 3rd Row : C. S. Seshadri, P. V. Abdur Rahman, S. K. Lakshmana Rao, K. Venkatachalaiengar, J. N. Dharap, S. Ramakrishnan, D. N. Abrol, P. V. Krishnayya, Chidambara Swamy, P. V. Patel, M. V. Jambunathan, H. Gupta, B. S. Jain.

Standing 4th Row : V. N. Patwari, V. Venugopal Rao, P. K. Raman, K. G. Ramanathan, L. C. Gupta, D. R. Kaprekar, Student, Student, S. K. Hindi, S. C. Saksena, M. K. Singal, Shanti Narayan, J. N. Kapoor, B. B. Bagi.

PROGRAMME

Thursday, December 23, 1954

11.00 A.M.	Address of Welcome by Dr. S. Bhagavantam, Vice-Chancellor, Osmania University
	Inauguration of the Conference and the opening of the Exhibition by Sri Gopalrao Ekbote, Minister of Education, Hyderabad
	Annual Report by the Secretary, Professor S. Mahadevan
	Presidential Address by Professor Ram Behari, University of Delhi
	Award of the Narasinga Rao Medal
2.00 P.M.	Meeting of the Council
2.30 P.M.	General Meeting
3.00 P.M.	Visit to the Exhibition
4.45 P.M.	At Home by the Reception Committee
6.00 P.M.	Popular Lecture
8.30 P.M.	Musical Entertainment

Friday, December 24, 1954

9.00 A.M.	Session for the presentation of papers
11.30 A.M.	Invited Address by Professor H. Rademacher, Tata Institute of Fundamental Research and the University of Pennsylvania, on the 'Theory of Partitions'
12.30 P.M.	Half-hour Address by Mr. B. V. Singbal, Tata Institute of Fundamental Research, on the 'Algebraic Basis of Morse's Variational Theory'
2.30 P.M.	Half-hour Address by Dr. R. P. Bambah, Punjab University, on the 'Geometry of Numbers'
3.00 P.M.	Symposium on the 'Teaching of Mathematics'

5.00 P.M.	At Home by the Vice-Chancellor, Osmania University
6.00 P.M.	Popular Lecture by Dr. P. L. Bhatnagar, St. Stephens College, Delhi, on 'Space Travel'
9.00 P.M.	Entertainment

Saturday, December 25, 1954

9.00 A.M.	Session for the presentation of papers
11.30 A.M.	Invited Address by Professor H. Maass, Tata Institute of Fundamental Research and the University of Heidelberg, on 'Symplectic Geometry'
12.30 P.M.	Half-hour Address by Mr. T. P. Srinivasan, Tata Institute of Fundamental Research, on 'Extensions of Measures'
2.30 P.M.	Half-hour Address by Professor S. M. Shah, Muslim University, Aligarh, on 'Exceptional Values of Entire and Meromorphic Functions'
5.00 P.M.	Reception by the Municipal Corporations of Hyderabad and Secunderabad
9.00 P.M.	Visit to the Nizamiah Observatory

Sunday, December 26, 1954

Whole-day	Excursion.

REPORT OF THE CONFERENCE

Inauguration

The Twentieth Conference of the Indian Mathematical Society was held under the auspices of the Osmania University, Hyderabad, Thursday through Saturday, December 23-25, 1954. It was attended by nearly a hundred delegates from all parts of India.

The Conference opened on the morning of December 23 in the Arts College Building with an address of welcome by Dr. S. Bhagavantam, Vice-Chancellor of the Osmania University and Chairman of the Reception Committee. The Vice-Chancellor read out a message from the Chief Minister of Hyderabad, wishing the Conference success. The Conference was then formally inaugurated by Mr. Gopal Rao Ekbote, Minister for Education and Local Government, Hyderabad. In his inaugural address the Minister pointed out how " without a knowledge of mathematics, we cannot plan the rational society in which there will be leisure for all and poverty for none ". He then referred to the contribution made by India to mathematics both in ancient times and more recently, the latter especially in the work of the late Srinivasa Ramanujan, and expressed the hope that the Conference would aid in the furtherance of mathematical study and research in India. He also declared open the Mathematical Exhibition which was housed in the Physics Building.

Report for the year 1953-54

Professor S. Mahadevan, Secretary of the Society, then presented the report for the year 1953-54. He conveyed the Society's thanks to the authorities of the Osmania University for their kind invitation to hold the Conference in Hyderabad. The Society had met in Hyderabad once before, in 1939, and was glad to meet again in that historic city to which it had special reasons to be attached, in as much as the great Bhaskaracharya hailed from the Bidar District of Hyderabad State.

The Secretary announced with deep regret the loss sustained by the Society in the death of Professor A. N. Singh of the University of Lucknow who was well-known for his work on Hindu Mathematics. Professor Singh was a valued member of the Society and an active member of the Council for many years.

The Secretary then recalled the activities of the Society during the year under report. He said, " we continue to publish both the periodicals, the Journal of the Indian Mathematical Society and the Mathematics Student. The Journal contains advanced research papers and has improved vastly in content and printing and get-up, improvements which are so necessary if we wish to conform to international standards. These improvements are due to the untiring energy of Professor K. Chandrasekharan of the Tata Institute of Fundamental Research, and to the enterprise of the Commercial Printing Press, Bombay, which is always ready to adopt the latest style and technique. The papers sent to the Journal are as usual refereed by experts.

" The other periodical, the Mathematics Student, is also under the charge of the editor of the Journal who is trying to improve its tone so as to make it serve as a real link between advanced research and the activities of young researchers and college teachers. We wish to see this periodical subscribed to by every college so as to be easily accessible to all students of mathematics. To make this popular we have priced it very low and hope that all students will take advantage of this offer. We wish to improve in particular, the section on class-room notes, problems and solutions, and also the news items of interest to all mathematicians. I request all teachers to contribute towards the improvement.

" Regarding the Library, which is located in the Ramanujan Institute of Mathematics at Madras, we are anxious that members should avail themselves of the facilities of circulation of current journals in which they may be interested. We are getting most of the foreign periodicals, either in exchange or by subscription.

With a view to improving the book section we have recently ordered a number of advanced books and have approached the Government for a liberal grant.

"Hitherto we were holding our Conference once in two years. But many wanted an annual gathering and now we are meeting once a year much to the satisfaction of our members.... In this connection I wish to thank the Government of India for giving travel concessions which we have been pressing for, for some years. I hope that our Society will be included permanently in the list of cultural bodies entitled to this concession".

The Secretary next referred to the participation of some of its members in international gatherings. He said, "We are glad that the Government of India deputed two members of the Society, Professor K. Chandrasekharan and Professor B. R. Seth, to attend the International Congress of Mathematicians at Amsterdam in September 1954. The Government also deputed Professor K. Chandrasekharan as the representative to the International Mathematical Union at the Hague. I am glad to state that Professor K. Chandrasekharan has been elected to the Executive Committee of the International Union for a term of four years, and that Dr. Ram Behari has been elected a member of the International Commission for Mathematical Instruction. I am also glad to note that Dr. B. N. Prasad of the University of Allahabad was chosen as a delegate to attend a recent meeting of the UNESCO in Montevideo."

Regarding the finances of the Society the Report said, "Our expenditure on printing and the Library is increasing, while our income derived from members and subscribers is more or less stationary. I wish to appeal to the Government of India to come to our help to put the Library on a firm footing. I wish to thank the Universities of Madras, Bombay and Osmania, the Tata Institute of Fundamental Research, and the National Institute of Sciences of India for their generous annual grants. I wish to thank the Government of India for their special grant of Rs. 2000. I appeal to other Universities to help us by grants.".

Referring to the Independence Day Awards, the Secretary said the Society was proud that Sri C. V. Raman, an honorary member of the Society, was the recipient of the highest award of Bharata Ratna, while two of the life-members of the Society, Professors H. J. Bhabha and K. S. Krishnan had been awarded the Padma Bhushan.

The Secretary extended a hearty welcome to Professor Hans Rademacher of the University of Pennsylvania and Professor Hans Maass of the University of Heidelberg who had so readily responded to the Society's invitation to attend the Conference.

Presidential Address

The President of the Society, Dr. Ram Behari, then delivered his presidential address which is printed separately.

Award of the Narasinga Rao Medal

On the conclusion of his address Dr. Ram Behari announced that it had been decided to award the Narasinga Rao Gold Medal for the previous year to Shri M. S. Ramanujan. At the President's request Mr. Ekbote presented the medal to Shri Ramanujan.

Business Meeting of the Society

A meeting of the Council of the Society was held in the afternoon, followed by a meeting of the General Body of the Society at which a resolution touching the death of Professor A. N. Singh was passed.

Proceedings of the Conference

Two sessions of the Conference were devoted to the reading of papers, on December 24, 25. Abstracts of these appear separately. On December 24, Professor Hans Rademacher of the University of Pennsylvania delivered, by special invitation, a one hour address on the 'Theory of Partitions'. There were two invited half-hour addresses

one by Mr. B. V. Singbal on 'The algebraic basis of Morse's variational theory', and another by Dr. R. P. Bambah on 'Geometry of numbers.' The invited hour-address on December 25 was delivered by Professor Hans Maass of the University of Heidelberg; his subject was 'Symplectic geometry'. It was followed by a half-hour address by Mr. T. P. Srinivasan on 'Extensions of measures' and another by Professor S. M. Shah on 'Exceptional values of meromorphic and entire functions'. On December 24 a Symposium was held on the 'Teaching of Mathematics'. It was presided over by Dr. Ram Behari, and Professors H. Rademacher, A. Narasinga Rao and K. R. Gunjikar, among others, participated. Two popular lectures were arranged; one on December 24 by Dr. A. Narasinga Rao on 'Some unusual applications of mathematics', and the other on December 25 by Dr. P. L. Bhatnagar on 'Space-travel.'

At the concluding session of the Conference the President expressed the sincere appreciation of all the delegates to the Osmania University and the Reception Committee for their warm hospitality.

Social Programme

The Entertainment of the delegates included receptions and a variety show. The delegates were housed in the University Hostel. The Reception Committee was at home to the delegates on December 23, and the Vice-Chancellor on December 24. A variety show was arranged on the evening of December 24. A Civic Reception to meet the delegates was held by the Corporations of Hyderabad and Secunderabad, and an address of welcome was presented by the Mayors of the two Corporations to which the President, Dr. Ram Behari, replied suitably. The delegates were taken round on a visit to The Nizamiah Observatory, Begumpet, on the evening of December 25. There was a full day excursion on December 26 to the historic fort of Golconda and other places of interest round about, in which several delegates joined. Another trip was arranged to Ajanta and Ellora which was taken advantage of by some delegates.

PRESIDENTIAL ADDRESS

By RAM BEHARI

We met in this magnificent historic city in 1939 and are meeting again today on the invitation of the Osmania University through its Vice-Chancellor, Prof. S. Bhagawantam, one of the most distinguished physicists of our country. I express our Society's grateful thanks to our hosts for their kind invitation.

Our Society was founded in 1907, by the late V. Ramaswami Aiyer for the advancement of higher mathematics and research. During the last 47 years, the Society has grown in strength and fame and has contributed a great deal to the progress of Mathematical research in our country. With our Society will always remain associated the name of the late S. Ramanujan, the greatest mathematical genius of modern India and a world Mathematician of the highest order. One of the main activities of our society has been the holding of biennial conferences at various centres of learning in the country and it is most gratifying that during the last three years the output of creative ability and research work by our members has increased to such an extent that it has become possible and desirable to hold our conferences annually rather than biennially. More than 50 original papers have been contributed to the present Conference. I hope our conferences will continue to be yearly in future so that we may keep abreast of what is happening in the world. In the experience of mathematicians, personal associations through correspondence and at individual and group meetings are the most fruitful for the development of mathematics. These conferences give us an opportunity of exchanging ideas, discussing current research, and of chalking out a few directions in which further advances could be made. Though the main concern of our Society is advancement of higher mathematics and research, yet teaching and research must not be separated. Without research, teachers will be reduced to the status of retail sellers as compared to producers, while without sound teaching of the fundamentals

of the subject, by able teachers who have mastery of the subject, research may not be possible except in the case of a few born geniuses.

During the last 20 years, the number of mathematical institutions and of mathematicians, has increased considerably in the world. There are about 5000 mathematicians who contribute original papers to mathematical periodicals.

Before 1930, it was very rare that a mathematician found employment in jobs other than teaching, research, and insurance but now there are innumerable fields, industrial, commercial and cultural in which there is a great demand of persons possessing mathematical skill. Some of the fields which hold greatest promises for jobs for those who have training in Mathematics are industry, factories developing the automatically sequenced electronic computers, operational research laboratories, the aircraft industry etc. In calculating machines also the role of mathematicians is no less important for it is they who prepare the programme and the material so that the machine may be used to give the desired results.

The number of books and periodicals disseminating mathematical knowledge is also constantly increasing. Thus, for example, in the Mathematical Reviews or the Zentralblatt für mathematik one can count about 900 periodicals which publish original mathematical papers. The number of periodicals devoted to elementary and intermediate mathematics also is very large. Almost every country possesses universities, science academies and scientific societies, producing and publishing original or informative mathematical results.

Mathematics plays a vital role in the growth of our industrial civilization. Indeed the capacity for national growth and development can be gauged rather accurately by the quality and extent of the national activity in mathematics. Advances in mathematical research are essential to advances in pure science, and hence to advances in technology. Everyone knows that we are living in a

technological era, but it is not often realized that our technology is based on Science, and therefore Mathematics, the Queen of Sciences, is bound to play a fundamental role. Mathematics is the basis of all the Natural Sciences with which it is connected in an indissoluble manner, but it also plays an important role in social sciences.

There are several fairly distinct branches of social sciences to which mathematics has been or may be applied. The chief of these may be distinguished as *Economics* and *Statistics*. In Economics, familiarity with mathematics will save many confusions of thought and will help in making clear the concept of economic equilibrium in which many factors act and react on each other.

As for *Statistics*, it has benefited very considerably by the entry of mathematicians in its field. Generally the same statistical problem arises in various fields but the statistician tries to solve it in terms of the special situation in which it arises without any effort to find a general solution. The mathematician, on the other hand, goes to the essence of the problem and tries to find a general solution which will apply equally to all fields. Advanced mathematicial methods have succeeded in solving many statistical problems, which were beyond the resources of applied statisticians. Knowledge of the algebra of determinants and matrices and of linear and quadratic forms, forms the mathematical ground work for the method of least squares and for much of the development in multivariate analysis, Stochastic Processes and Econometrics. Advanced calculus, particularly multiple integration and expansion in series, a course on complex variable, combinatory analysis, vector analysis and n-dimensional geometry, are useful in dealing with distribution problems. Point-set and measure theory are useful for following some of the more fundamental developments in probability; finite groups has recently found applications to the theory of the design of experiments; calculus of variations is the basis of the general theory of testing hypotheses. Numerical methods including interpolation, numerical integration, and finite differences are useful

because the results of theoretical investigations generally require the construction of tables before they can be handled by the non-mathematical applied worker. It is hoped that able young mathematicians will continue to be attracted into this field also which offers a wide range of useful and stimulating applications. It is no exaggeration if I say that every major concern among the intellectual concerns of man is a concern of Mathematics. The world we live in is dynamic and not static. *Change* is the great Master that rules us all. Mathematics consists mainly in the study of function and the study of functions is the study of the ways in which changes in one or more things produce changes in others. The famous mathematician *Klein* once said, " the world is being functionally minded and without a notion of functionality it is not possible for any one to understand modern civilization and its trends. " The study of advanced mathematics is important not simply because of its utilitarian value in the narrow sense as a master tool for scientific inventions but chiefly because in our modern civilization it has to be regarded as one of the Humanities, the study of which is indispensable for a rational life for it gives training in abstraction, logical consisteney, precision, economy of thought, clarity and exactness, and inculcates the habit of correct thinking. Mathematics is an Arts subject for it provides an outlet for the creative instinct of man. It is at the same time the basis of all the science subjects. So it stands as a symbol for the unity of the human mind. It is very gratifying that the Central Government has recognised this fact and has included Mathematics in the list of subjects under Humanities also for the award of research scholarships.

The importance of mathematics and through it of the mathematician in a wider sense is growing in proportion to the development of mankind; new possibilities, new opportunities for applying mathematics are discovered. The extraordinary success obtained in the exploration of inert and living nature both in small and large infinite dimensions, the theoretical and technical discoveries with their enormous possibilities of application, the newly acquired perspectives in social and human sciences have not ceased enriching and

extending considerably the field of *abstract* and *applied mathematics*. Among some of the most recent extensions of mathematics may be mentioned Biometry, Econometry and Cybernetics. Matrix methods are found to be essential in modern econometrics. Free discussions between specialized different fields, in particular, between mathematicians, engineers and biologists have given rise to Cybernetics, which gives account of the synthesis of calculations, diagrams and machines.

There is a proposal before the International Commission on Mathematical Instruction for the production of a film about the role of mathematics and mathematicians at the present time. The object of the film is to show the universal role of mathematics, i.e. to show that the tremendous importance of mathematics is present everywhere, where mankind is active. The film would show by pictures, forms, movements, action, music, etc. the presence of mathematics in every national phenomenon as well as in every spiritual activity. The film is intended to represent various forms of matter, life, social phenomena, arts, techniques, robots, infinitesimal and transfinite considerations etc. with tendency to show mathematics and mathematicians respectively as very appropriate links between various fields of activity and various countries, to contribute to the social progress and mutual understanding of thinking individuals and groups respectively. I have no doubt that such a film will be a great document for future generations; it will be a stimulus for the development and diffusion of mathematical thought and the adequate consideration of mathematics in its various interactions with other human activities. I also wish the International Mathematical Union every success in its undertaking of publishing a World Directory of Mathematicians.

It is usual and customary at our Conferences to take stock of the position of teaching and research in mathematics in the country and to suggest necessary steps for improvement. We find that there is a vast reservoir of mathematical talent in the country. Whenever proper guidance and financial help in the form of scholarships

of decent amounts and reasonable duration are given, the achievements are very encouraging. The number of students in the post-graduate classes in mathematics in our universities has increased considerably in the last few years. In some universities where the number used to be even below 10, it is now in the neighbourhood of 150. The number of research students also has grown considerably and is fast increasing year after year. The out-put of creative ability also is very gratifying. There are more than a dozen journals in which original papers on mathematics are published. The quality and size of our own journals viz. the *Journal of the Indian Mathematical Society* and the *Mathematics Student* have very much improved. Besides Universities, research work is done in some institutes devoted exclusively to scientific research, viz. The Tata Institute of Fundamental Research at Bombay, Sir C. V. Raman Institute at Bangalore, The Statistical Institute at Calcutta and Ramanujan Institute at Madras. Research work in Astronomy is done at the Kodai-kanal Observatory and the Nizamia Observatory, Secunderabad. A new observatory has recently been set up at Banaras. Several Universities have started facilities for imparting instruction and research in Ballistics which is important in Defence Science. I would, however, like to point out that when we look to the vastness of size and population of our country, we find that a lot in this direction is still to be done to restore our country to the pristine honoured role of intellectual leadership in mathematics in the world which it enjoyed till the 15th or 16th century. It may be mentioned that the decimal place value system in arithmetic, the invention of zero, the fundamentals of indeterminate analysis in algebra, the sine and cosine functions and the series now known as Gregory's series in trigonometry, and various other well-known theorems have been given to the world by ancient Indian mathematicians, the prominent among them being Aryabhatta, Brahmagupta, Mahavira, Sridhara and Bhaskara.

For the benefit of our young research workers, I would like to take this opportunity of mentioning some of the present tendencies in mathematics, i.e. the directions towards which modern mathematics

is drifting. Methods in Topology, Algebraic Geometry, Finite and Discontinuous Groups, representation of Lie's Groups in Banach spaces, Holonomy and Homotopy as applied to Global Differential Geometry, a Differential Calculus free from the bondage of real axis, Theory of Quasi-periodical functions, statistical view point borrowing its methods from the recent theory of aleatory functions and methods in Mathematical logic, methods of continual approach in Fluid mechanics and Statistical mechanics of Turbulence, and Calculating machines may be mentioned as some new approaches which are likely to yield a rich harvest of new results. If the country is to regain its former position, more importance and recognition should be given to mathematical research and creative out-put.

The following are some of the causes which retard the progress of Mathematical research in the country :

1. Outmoded syllabi still being generally followed.

2. Scarcity of inspiring teachers in colleges and universities where sometimes research qualifications are not given their due importance at the time of selection.

3. Lack of research facilities in many institutions and uncertainty of getting suitable careers after completing a research course.

4. Comparatively low scales of pay in most of our educational institutions.

These draw-backs can be remedied in the following ways :

1. Teachers should consider it their duty to discover talent in their pupils and to stimulate a passion for mathematical research. They should dedicate themselves to their work, to the search for truth, and to the discovery of the secrets of the universe for the good of mankind.

2. In the context of present day requirements, courses should be reinforced with additional subjects, comprising applied aspects.

Regular seminar activities consisting of lectures and group discussions on advanced topics and current research should be organised

so that students may come in contact with research workers and keep themselves informed about recent developments.

3. Research facilities in the form of scholarships, and libraries well equipped with modern books and journals should be provided.

Publication of monographs on advanced topics should be encouraged. Financial and technical difficulties which obstruct the prompt publication of research papers should be removed.

4. Instead of having lonely and stray workers scattered about, efforts must be made to build up schools of different branches of mathematics so that at different centres team work may be possible with mutual study and discussion. Corporate activity of pooling the resources of several universities could also greatly facilitate research.

5. Change in the present outlook and sense of values is necessary. More importance and recognition should be given to creative work. To illustrate my point I may mention an anecdote. In Germany, in 1890, everyone and everything connected with university was held in the greatest popular esteem. A philosopher at Heidelberg was one day disturbed in his studies by the noise of workers in the street before his house. He had, at that time, been offered a professorship in Berlin. He opened his window and shouted to the workmen, " If you do not stop that noise at once, I will accept the call to Berlin ", whereupon, the foreman ran to the mayor who postponed repair of the street until after the coming of the academic vacation.

Mathematical study requires passionate hard digging and there is no royal road that materially shortens the process. The persons engaged in it have to be far from the focus of public interest, neither much encouraged by the applause of the masses nor disturbed by their criticism. Mathematical ability must be considered as an artistic ability and must be patronised in the same way as the work of painters, artists and sculptors. It is true that the creators of mathematics are motivated by the love for creation ; they experience delight when they succeed in formulating their ideas and building up ingenious and masterful proofs. The reward of a

mathematician is the joy he derives from the value and harmonious beauty of the treasure of the mathematical results he obtains and the pleasure and exhilaration of successful thinking. But after all mathematicians are also human beings and must have means which are at least sufficient to sustain them and to enable them to meet the bare necessities of their dependants.

Sir Shanti Swarup Bhatnagar, in his address which he delivered at the Convocation of Rajputana University held in November, 1954, rightly remarked, " The frustration of the teacher was reflected in his work and a dissatisfied teacher could not be expected to yield an healthy influence. If the teacher was not provided with his minimum material needs he could not be expected to maintain his interest in teaching and scholarship. While standards and cost of living had risen enormously, there had been practically little change in the salaries of teachers except in rare colleges ......Salaries of staff should be such as would attract competent teachers."

I would like to draw the attention of the delegates who have assembled here from different parts of the country to two important problems which need deliberation :

The first problem is to consider the changes to be effected in the teaching methods and teaching materials which have become necessary on account of several new branches of mathematics which have recently been developed and on account of the tremendous applications of mathematics to industrial, commercial and cultural enterprises and the consequent increase in the number of students taking up mathematical study. The problem of instruction in mathematics has to be examined in the light of the social scientists' needs quite as much as those of the natural scientists and engineers. It has to be brought in tune with the spirit and needs of the times.

The other important problem which is facing us and which needs immediate attention is the language problem so far as advanced mathematics and research are concerned. While considering this

question, it should be kept in mind that mathematics is international and that mathematical methods are the same all the world over: a mathematician deals with essential relations, irrespective of the country, religion, race and nationality to which he belongs ; his role is to determine what is essential in a situation and to contribute to the mutual understanding of individuals or groups of individuals or domains of arts and sciences.

I now pass on to another phase of this address and by considering the history and growth of Geometry in which we come across an outstanding example of purely abstract concepts and concrete facts of experience being intimately related to each other, I shall try to show how new mathematical ideas have brought a revolution in modern philosophy and how mathematics has played a predominant role in the formation of culture. We shall see incidentally how mathematics helps in fulfilling the aim of science, viz. to describe and explain the world we live in, how it explains the minute world inside the atom and the phenomena involving great distances, times and velocities, or in other words how it unites the microscopic and macroscopic worlds by series of mutually consistent equations. We shall see how much Physics has become mathematised in the form of Geometry.

The early Babylonians, Egyptians, Indians, Chinese and Japanese, who may be called the ancestors of modern mathematics, took interest in geometry but they concerned themselves with matters of practical need only. This early geometry consisted of a number of crude rules for the mensuration of various simple geometrical figures. The ancient Greeks developed this crude beginning into the Science which is now studied in schools as demonstrative geometry comprising the plane and solid geometry—the plane geometry dealing with the properties of figures in a plane and the solid geometry dealing with the properties of figures in space of three dimensions. This form of geometry depends on the observation that the propositions of geometry are logically inter-related, i.e. if certain propositions are granted as axioms and postulates, others can be proved as

logical consequences of those assumed. The first comprehensive and systematic attempt to exhibit the propositions of geometry in a logical sequence was the 'Elements of Euclid' written between 330 and 320 B.C. The Greeks held geometry in great esteem. Plato placed the inscription in his porch "Let no one who is unacquainted with geometry enter here." He sought in geometry the key to the universe. When asked about the occupation of God, he answered, "He geometrises continually".

A new stimulus came to the development of geometry by the introduction, in 1637 A.D., by Descartes, of the so called Analytic geometry by which properties of geometrical figures came to be discussed by algebraic equations and not by the figures themselves. The introduction of the differential and integral calculus, in 1687, by Newton and Leibnitz, greatly increased the power of Analytical geometry by the ideas like curvature and torsion and gave rise to differential geometry. Euler, Monge, Gauss and others developed differential geometry of curves and surfaces.

A great change in the nature of geometry came in the 19th century by the invention of the non-Euclidean geometries. Euclidian geometry is founded on ten axioms, viz.; it shall be possible to draw a straight line through two given points ; a terminated straight line may be extended without limit in either direction ; it shall be possible to draw a circle with given centre and through a given point; all right angles are equal; through a point P not in a line L one and only one straight line can be drawn which does not intersect L; things equal to the same thing are equal to one another; if equals are added to equals, the results are equal; if equals are subtracted from equals, the remainders are equal : the whole is greater than any one of its parts ; things that coincide are equal. Ever since the time of Euclid, geometricians had tried to deduce the fifth axiom, viz. the parrallel postulate, from the other postulates of Euclid's system but did not succeed. About 4000 books or papers were written by about 1350 authors to prove it but to no success. Finally, three mathematicians, Gauss, Lobachevsky and Bolyai working inde-

pendently argued that if it is impossible to prove the parallel axiom from others, it must be independent of these others and can be therefore replaced by a different axiom. These three replaced the parallel axiom by the axiom that through a point P not in the line L, an infinite number of lines parallel to L can be drawn. With this new axiom and the other nine of Euclid's axioms they built up a geometry which is called Hyperbolic geometry which can be pictured on a pseudo-sphere just as Euclidean geometry can be pictured on a plane. In Euclidean geometry one theorem is that the sum of the three angles of a triangle is equal to 180° but in Hyperbolic geometry the sum of the three angles of a triangle is less than 180°. Gauss applied this triangle test to determine which geometry fitted better with physical space. He stationed three observers, one on each mountain peak, and measured the angles of the triangle thus formed by light signals. He found that the sum was within 2″ of 180° and gave up the experiment as indecisive. He obviously made two mistakes. The first mistake was that the triangle he chose was not large enough and the second mistake was that he assumed rays to travel in straight lines which is not so. Another great mathematician Riemann (1826-1866) replaced the parallel axiom by the axiom that through a point P not in the line L, there is no line which will not meet L, i.e. there is no line through P which is parallel to L. He got the novel idea that a distinction must be made between endlessness and infiniteness and that Euclid's second axiom, viz. that a "terminated straight line can be extended without limit in either direction" must be replaced by the axiom that "a straight line is endless but finite". He also conceived that "two points may determine more than one line", whereas Euclid's first axiom suggested only one such line. The geometry which Riemann built up is called Riemannian geometry or Elliptic geometry and may be pictured on the sphere. The straight line in the sense of Euclidean geometry has no existence at all on the surface of a sphere. It appears, therefore, that for us who are living on the surface of earth, the natural geometry is Riemann's geometry and not Euclid's geometry. Gauss, Lobachevsky, Bolyai and Riemann asserted that

there are geometries different from Euclidean geometry but equally valid. Geometry thus came to be regarded as a formal hypothetical deductive exercise in logic and became axiomatic in nature. Man could construct a new geometry and decide to view space in its terms. Cantor's assertion that 'the essence of mathematics is its freedom' was realised. The main contribution of non-Euclidean geometry to mathematical thought as a whole and also perhaps the principal contribution of mathematics to the progress of civilization is the deflation of older beliefs including the abolition of mathematical absolutism. It brought a revolution in philosophical thought. Till about 1850, philosophy was based on the unchallangeable truth of Euclidean geometry. Kant, the great German philosopher, declared that not only is Euclidean geometry a true description of space but that this is the only description which human beings can ever hope to give since they are incapable of thinking of space in other than Euclidean way. The invention of non-Euclidean geometries showed that Kant was wrong. Euclidean geometry is only one of many possible ways in which human beings can think of space. Each non-Euclidean geometry is an alternative way. The new geometries provide a good example of the power of the mind which can reject the evidence of the senses for the sake of what the mind might produce. Non-Euclidean geometry made possible a new point of view in physical science, viz. that scientists might be able to offer better theories to describe the physical world if they used non-Euclidean geometry instead of Euclidean.

Einstein comes to our mind as an outstanding example of a physicist who adopted this new procedure and built up his theory of Relativity which is one of the most beautiful chapters in science and is one of the two most brilliant achievements of the 20th century, the other being Quantum mechanics. The space and geometry used by Einstein in his theory of Relativity are Riemannian and not Euclidean. In 1905 Einstein inaugurated a series of sweeping changes that were needed to re-establish physical theory and propounded the Special theory of Relativity. The Michelson-Morley experiment and other experiments of this nature showed that it was not possible

to find the velocity of a material body in ether assuming the latter to be stationary. Einstein adopted the basic assumption that the velocity of light is the same for all observers in the universe irrespective of the manner in which they may be moving relative to each other. He also accepted as an axiom that no physical body can have a velocity greater than the velocity of light. Einstein discarded Newton's notions of 'absolute length' and 'absolute time' and introduced the concepts of 'local length' and 'local time'. According to him, two observers who are in motion relative to each other will obtain different measurements of distance and of time between the same two events. Thus for example, an observer in moon will measure the distance and time between the same two events differently from an observer on the earth.

Einstein also discarded absolute laws. He concluded that the laws formulated by two observers in terms of their own lengths and times can be transformed, one into the other, by means of formulae connecting their length and time measurements and their relative velocity. In 1908, Minkowski pointed out that the natural view of events should be in terms of a combination of space and time, i.e. the world is a four-dimensional space-time continuum, for time has been measured by spatial means, e.g. in terms of the distance moved by the hands of a clock, by the motion of a pendulum through space or by the distance a shadow travels on a sundial. Also, distance has been measured by applying a rod, but when we do so, time elapses, as no measurement is instantaneous. Nature presents space and time together. Einstein utilized Minkowski's view of regarding space and time as a four dimensional unity in his General theory of Relativity, which he propounded in 1915.

Einstein assumed that ds, the space-time interval between two events is the same for all observers, i.e. ds is an invariant. The form of ds determines the character of geometry to be used. It determines also the geodesics, i.e. the curves which give the shortest distance between two points. Einstein's brilliant idea was to choose an expression for ds so that the path of each planet is a geodesic in

the resulting geometry. Just as Newton introduced the notion of a force of gravity to account for the fact that planets move in ellipses round the sun in a focus, instead of in a straight line was warranted by his first law of motion, so by assuming that material bodies moved in geodesics, Einstein did away with the need of introducing a fictitious force of gravity.

In Newton's theory, time and length are the same for all observers in the universe; there is a force of gravity which reaches us from the sun so far away and keeps the planets in their orbits. But in Einstein's theory, our physical world is a four dimensional mathematical continuum in which gravity, time and matter along with space become merely part of the structure of geometry; the character of the geometry and the shape of the geodesics are affected by the presence of masses such as the earth or the sun. The theory of Relativity is an example of a mathematical creation that has revolutionized modern philosophy. Two predictions of the theory of Relativity are now indispensable to science, viz. relativity of mass, and equivalence of mass and energy.

Before the advent of the theory of Relativity physics had two conservation laws of fundamental importance, viz. the law of the conservation of mass and the law of the conservation of energy which appeared to be quite independent of each other. The theory of Relativity unites these into one law. According to the theory of Relativity the energy is equal to the product of mass and square of the velocity of light. This is one of the most famous equations in history. Einstein showed that mass and energy are equivalent, the property called mass is simply concentrated energy. Matter is energy and energy is matter, the distinction is one of temporary state. This equation explains how the sun and stars go on radiating light and heat for billions of years. This equation has played an important part in the development of the atomic bomb. It forecasts how many grains of uranium must go into a bomb in order to destroy a city. It explains how radioactive substances like radium and uranium are able to eject particles at enormous velocities and go on doing so for millions of years.

The Special or the Restricted theory of Relativity is concerned with the invariant formulation of physical laws subject to the restrictions that observers move with uniform relative velocity and that gravitation is absent. In the General theory of Relativity both these restrictions are removed. Einstein takes the invariant ds^2 as $g_{ij}\, dx^i\, dx^j$ where g's are functions of the coordinates. The fact that g's are not constants but functions of the coordinates indicates the departure from the non-inertial character of the frame of reference; it indicates the presence of gravitation and g's may be regarded as the gravitational potentials. The free path of a particle is given in special theory by the variational principle $\delta\int ds = 0$. Since ds is invariant, it holds for any accelerated frame of reference also. Thus the free paths of material particles are geodesics and the light rays are the null geodesics. The Christoffel symbols are analogous to the gravitational force. For empty space the law of gravitation is taken to be $R_{ij} \equiv R^k_{ijk} = 0$, where R_{ij} is Ricci's tensor, which corresponds to Laplace's equation; and in the space containing matter it takes the form $R_{ij} - \frac{1}{2} R g_{ij} = -8\pi\, T_{ij}$, where $R \equiv g^{ij} R_{ij}$ and T_{ij} is the energy momentum tensor, which corresponds to Poisson's equation.

It may be pointed out that Einstein himself was not satisfied with the character of his field equation $R_{ij} = \frac{1}{2} R g_{ij} = -8\pi\, T_{ij}$. He compared the left hand side of this equation, which depends upon g_{ij} with a pillar of marble and the right hand side which depends upon the enegry momentum tensor, with a pillar of straw. Taking Schwarzchild's form of the metric, viz.

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \\ + \left(1 - \frac{2m}{r}\right) dt^2,$$

he verifies the three crucial tests satisfactorily. The advance of the perihelion of Mercury by his theory comes out to be 42.9″ per century which nearly agrees with the observed value of 43.5″. His theory shows that a ray of light coming from a star grazing the sun's surface will suffer a deflection from its path, a phenomenon

which was not known before. The amount of deflection calculated from his theory comes to be 1.75″ which nearly agrees with the values observed at the times of the solar eclipses of 1919 and 1922. The red shifts in the spectrum of the sun and in that of the companion of Sirius, as predicted by Einstein's theory, are also nearly the same as those observed by astronomers.

Besides these three crucial tests, however, no other phenomenon in verification of Einstein's theory has yet been suggested. While Einstein's General theory of Relativity appears to give a satisfactory account of gravitation, it does not take electromagnetism into account at all. The problem, therefore, is to find field equations which would explain not only gravitational phenomena but also electromagnetic phenomena, i.e. to find a unified field theory of gravitational and electromagnetic phenomena.

To explain the unseen realm of the atom and the fathomless depths of intergalactic space, i.e. to explain the realities that are too small to be perceived and the realities that are too remote and too vast to be perceived, two great theories were developed between 1900 and 1927 : (i) Quantum theory dealing with the fundamental units of matter and energy, and (ii) Relativity theory dealing with space, time and the structure of the universe as a whole. Yet these two great scientific theories rest on entirely different and unrelated theoretical foundations. Believing in the harmony and uniformity of nature and rejecting the idea that there may be two structures of space independent of each other, Einstein, after struggling for about 35 years, published in 1950, his generalized theory of gravitation in which he has tried to find a unified field theory to unite the laws of gravitation and the laws of electromagnetism within one basic superstructure of universal law. He gives in one series of mutually consistent equations the physical laws governing the two fundamental forces of the universe, the gravitational force and the electromagnetic force. He does this by changing the nature of his tensor g_{ij}, i.e. by considering g_{ij} to consist of two parts, a symmetric tensor $\bar{g}_{ij}$ and a skew-symmetric tensor $\hat{g}_{ij}$.

Just as relativity reduces gravitational force to a geometrical property of the space-time continuum, so does the unified field theory reduce electromagnetic force to a similar geometrical property. He has obtained a new set of field equations and has described them as a system of utmost simplicity. He asserts that the results obtained appear to be the natural extension of the General theory of Relativity. The search for a perfect field theory is still going on. No solution of the field equations free from singularities has yet been found. During the years 1951-53, Eisenhart has introduced the idea of a space with which is associated a non-symmetric tensor g_{ij}. He calls this space generalized Riemann space. With the help of the corresponding geometry he has obtained the same equations which Einstein has obtained in his unified field theory. Another differential geometer, V. Hlavaty, has opened a powerful attack on the recent unified field theories of Einstein and Schrödinger. The basic problem was to find the geometry implied by the 64 equations relating the affine connection to a fundamental quadratic tensor which is neither symmetric nor skew-symmetric. Hlavaty solved this problem by outright tensorial solution of the equations themselves.

A Japanese mathematician, Takasu, has developed a geometry which he calls the geometry of non-holonomic spaces with general affine connection. By using non-holonomic Laguerre geometry of the second kind, he has established a theory of gravitational waves in very simple form. Taking further Laguerre's space L_3 as a Cartesian space, he has established the Special Relativity theory and an interesting Intermediary Relativity theory. He has shown that the General Relativity is a three dimensional non-holonomic Laguerre geometry of the second kind. He has further shown that the special Relativistic quantum-mechanics is an ordinary three-dimensional Laguerre's geometry. He has obtained the general relativistic generalization of the Dirac equation and of the Schrödinger-Gordon equation, which were hitherto considered to be impossible. Takasu claims that more than a dozen unitary field theories hitherto known can be examined from the point of view advanced by him without making any assumptions. Deviating

slightly from Einstein's idea, Takasu claims that Physics is global geometry and that physical phenomena take place in our Cartesian space in which several branches of global geometry are realised, each branch describing a physical phenomenon, and that Physics is classified into branches corresponding to the branches of the geometry it is based upon.

Thus we see that in both the major inventions of the 20th century, viz. the theory of Relativity and the Quantum theory, more and more of complicated mathematics is being resorted to.

One is inclined to agree with Thoreau when he says that "the most distinct and beautiful statements of any truth must take at last the mathematical form", and with Whitehead when he says that "the fact that changes in our material universe can be predicted—that they are subject to mathematical law is the most significant thing about it, for mathematical law is the concept of the mind and from the existence of mathematical law we infer that our minds have access to something akin to themselves, i.e. in or behind the universe."

SYMPOSIUM ON THE TEACHING OF MATHEMATICS

President : PROFESSOR RAM BEHARI

Professor Ram Behari informed that sometime back a committee was formed to standardise the syllabi but nothing definite was done and that the International Mathematical Union had formed an International Commission on Mathematical Instruction. He suggested that the discussion be confined to the following points :—

1. What students should be admitted to the Universities ? (condition of eligibility for admission).

2. Branches to be taught at different stages viz. high school or higher secondary stage ; graduate stage ; post-graduate stage.

3. Selection of teachers in Colleges and Universities.

4. System of examinations.

5. Methods of instruction for large classes.

6. Whether the syllabi in Universities should be in agreement with those in the competitive examinations.

Professor Ellis remarked that the subject matter now taught is very old. Students and instructors should both be interested in their subject. Exercises should be given to the students and day-to-day work should count. If the Society changes, then the whole organisation of teaching should come under review.

Professor Mahadevan : Training of teachers upto higher secondary and initial college stages is essential as these teachers mould the boys, The teachers are at fault if they cannot create interest in the boys.

Professor Rademacher informed that Allendoerfer Oakley have published a book on Teaching of Mathematics. He said that the capacity for thinking should be developed; basic ideas should be made clear and much time should not be spent on drill. Teaching does not necessarily gain from training of college lecturers, because a scholar can inspire students better.

Professor A. Narasinga Rao said that the general lack of scholarship and enthusiasm were responsible for the lowering of standards. The number of scholars was very small for a large country like India. Tata Institute of Fundamental Research and Ramanujan Institute were doing good work. Mathematics deals with order, rhythm etc. and methods of training will have to differ from those of other subjects. Teachers should know more mathematics too. The subject should be taught in a concrete way. Boys should frame examples and not only work them out. *The teacher should grow, the student should learn.* The teacher should take up new topics for study. New subjects may be taken up at particular stages but ideas should be given at all stages.

Professor Gunjikar said that facilities should be provided to students who have had no background. The content of the curriculum should change. Teachers should get more help at schools and colleges. Form should get more attention than content in training colleges. A little amount of training and guidance from seniors might be desirable. Topics that could be done away with should not be taken up and too much value should not be given to drill. The teacher should continue to learn. This will help him to understand the difficulties of students. The language question is also important and needs deliberation.

Professor M. V. Jambunathan : Teachers need to be interviewed at length before appointment. Training of teachers is helpful though not in research, but this training should come after selection.

Mr. K. T. Shah : The subject has to be taught at length in a systematic way, it is considered dull when it is not so taught.

Professor H. R. Gupta : A committee should be set up to implement the decisions arrived at.

Dr. Bambah : Things should be made to get going.

Mr. J. Willington (Public School, Hyderabad) : Interest should be created in the subject by the teachers.

Dr. Ram Behari, winding up, remarked that there should not be too much emphasis on drill, but emphasis should be laid on basic ideas. Teachers should grow, students should learn. The students should be encouraged to frame questions themselves rather than remain content with merely solving those framed by others; in this way they will get more insight into the fundamentals of the subject. Antidated and outmoded topics should be deleted from the courses of reading, and modern subjects should be introduced in their place, so that students may keep abreast with modern developments and may be able to proceed to research without much waste of time.

ABSTRACTS

ALGEBRA AND THEORY OF NUMBERS

K. G. RAMANATHAN, Bombay : *Symmetric and skew symmetric forms over division algebras*

Let D be a division algebra of finite rank over the field Γ of rationals. Let $x \to x^* \in D$ be an involutorial anti-automorphism of D/Γ. Let $M_n(D)$ denote the algebra of n rowed square matrices with elements in D. For any matrix $N \in M_n(D)$ let $N^* = (s^*_{lk})$, $N = (s_{kl})$. Then $N \to N^*$ is an involution of $M_n(D)$. If $N \in M_n(D)$ such that $N = N^*$ it shall be called *symmetric*; if $N = -N^*$ then N is called *skew symmetric*.

Let P be an order in D over Γ and let P_n denote the order of matrices N in $M_n(D)$ with elements in P. If N and N^{-1} both lie in P_n then N is called a *unit* of P_n. (P_n is a ring). The units of P_n form a group G with a finite number of generators (Siegel). Let $S \in M_n(D)$ which is symmetric or skew symmetric. A unit U is called a *unit of* S if $U^* SU = S$. (Norm of S over Γ is $\neq 0$). The units of S form a group $G(S)$. It is proved that $G(S)$ also has a finite set of generators.

In the special case of skew symmetric matrices a measure formula analogous to Siegel's for quadratic forms is also proved.

K. SUBBA RAO, Vizianagram : *On the recurring period of the reciprocal of an odd number.*

I prove, among other results, the following :

THEOREM 1. *If n is an odd number with a maximum recurring period $\phi(n)$ for $1/n$, then n must be of the form p^m, where $m \geqslant 1$ and p is a prime other than 5.*

THEOREM 2. *There are infinitely many odd numbers whose reciprocals have the maximum recurring period.*

D. R. KAPREKAR, Devlali : *Multidigital numbers*

Some properties of 'multidigital numbers' are proved.

D. R. KAPREKAR, Devlali : *Multigap numbers*

Some properties of 'multigap numbers' are proved.

HANSRAJ GUPTA, Hoshiarpur : *Partitions into distinct primes*

In this paper, I have given the first 301 coefficients in the expansions of $\Pi(1 - x^p)$ and $\Pi(1 + x^p)$ where p runs through primes 2, 3, 5, 7 etc. The first 201 coefficients in the latter expansion were obtained also by Om Prakash Gupta independently. The first table was required in order to check a table of partitions into primes recently computed by Mr. O. P. Gupta and Miss Luthra.

HANSRAJ GUPTA, Hoshiarpur : *Partitions in general*

In this paper, the number $P_r(n)$ of partitions of n into integers

$$a_0 = 1, a_1, a_2, a_3, \ldots, a_r,$$

where

$$a_0 < a_1 < a_2 < a_3 < \ldots < a_r,$$

is considered. From elementary considerations, it is shown that

$$\binom{n+r}{r} < P_r(n) \prod_{j=0}^{r} a_j < \binom{n+c_r}{r},$$

where $c_r = \sum_{j=0}^{r} a_j$. Hence, for sufficiently large n,

$$P_r(n) \sim \prod_{j=0}^{r} a_j^{-1} \binom{n+r}{r}.$$

These are generalizations of earlier results. Further investigations on the subject are still being carried on.

R. P. BAMBAH, Hoshiarpur : *Four squares and a k-th power*

Let $\lambda_1, \ldots, \lambda_4, \mu$ be non-zero real numbers, not all of the same sign and such that at least one ratio λ_r/λ_s is irrational. Suppose k is a positive integer. Then there exist arbitrarily large P such that the inequalities

$$1 \leqslant x_1, \ldots, x_4 \leqslant P^k, \ 1 \leqslant x_5 \leqslant P^2, \ |F(x)| = \left| \sum_{r=1}^{4} \lambda_r x_r^2 + \mu x_5^k \right| < 1,$$

have more than γP^{2k+2} integral solutions, where $\gamma = \gamma(\lambda_1, \ldots, \lambda_4, \mu) > 0$. As a corollary it follows that F can be made arbitrarily small in absolute value for integer values of the variables $x_1, \ldots, x_5$.

R. P. BAMBAH, Hoshiarpur : *A note on polar reciprocal convex bodies*

Let K and K' be two convex bodies in the n-dimensional Euclidean space which are polar reciprocal to each other with respect to the unit sphere (n-dimensional) centred at the origin O. Let $V(K)$, $V(K')$ and J_n be the volumes of K, K' and the unit sphere respectively.

If K and so also K', has O as its centre, then

$$V(K)\ V(K') \geqslant n^{-n/2} J_n^2. \qquad (1)$$

If K does not have O as its centre but contains it as an inner point, then

$$V(K)\ V(K') > 4^n/(n!)^2. \qquad (2)$$

The inequalities (1) and (2) are improvements on some theorems of Dvoretzky, Rogers and Mahler.

R. P. BAMBAH, Hoshiarpur and K. ROGERS, Cambridge. : *On non-homogeneous functions*

Let $f(x, y)$ be a function such that $f(x, y) \leqslant 1$ is a "region with hexagonal symmetry". Let

$$F(x, y) = f(\alpha x + \beta y, \gamma x + \delta y),$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary real numbers with $\alpha \delta - \beta \gamma \neq 0$. Then for any real numbers x_0, y_0 there exist integers x, y such that

$$F(x + x_0, y + y_0) \leqslant \max \ \{F(\tfrac{1}{2}, 0), F(0, \tfrac{1}{2}), F(\tfrac{1}{2}, \pm \tfrac{1}{2})\}.$$

If the above is true with strict inequality for $(x_0, y_0) = (\tfrac{1}{2}, 0), (0, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2})$, then it is true with strict inequality for all x_0, y_0.

This generalizes a result of Bambah on non-homogeneous binary cubic forms.

ANALYSIS

Kumari Girija Khanna, Aligarh : *On the integral functions defined by a Dirichlet series*

We define here the lower order (R) of the integral function $F(s)$ defined by the Dirichlet Series $\sum_0^\infty a_n e^{-s\lambda_n}$, $(s = \sigma + it)$, by the relation $\lambda = \liminf_{\sigma \to -\infty} \{\log \log M(\sigma)\} /(-\sigma,)$ where $M(\sigma) = \text{l. u. b.}_{-\infty < t < \infty} \ |F(\sigma + it)|$ and prove the following theorems :

Theorem 1. *Let* $\liminf_{n \to \infty} \frac{\log |a_n|}{\lambda \log \lambda_n} = c$, $(-\infty \leqslant c < 0)$; *and* $\log \lambda_n \sim \log \lambda_{n+1}$; *then* $\lambda \geqslant -1/c$.

Theorem 2. *Let* $F(s) = \sum_0^\infty a_n e^{-s\lambda_n}$ *be an integral function of order* (R) *equal to* ρ, *for* $\left(\sigma_A^F = -\infty\right)$, *and* $\liminf_{n \to 0} \frac{\lambda_n}{\log n} > 0$. *If* $F_1(s) = \sum_0^\infty b_n e^{-s\lambda_n}$ *be another integral function of order* (R) *equal to* $\rho(F_1)$ *and if* $|a_n| \geqslant |b_n|$, *then* $\rho \geqslant \rho(F_1)$.

Theorem 3. *If* $F(s) = \sum_0^\infty a_n e^{-s\lambda_n}$, *with* $\sigma_A^F = -\infty$, $\liminf_{n \to 0} (\lambda_{n+1} - \lambda_n) > 0$; *and if* λ_S *is the lower order* (R) *of* $F(s)$ *in each horizontal strip* S (πa) *with* $a > D$, *then*

$$\lambda_S \geqslant -1/c.$$

K. Balagangadharan, Bombay : *An extension of a Tauberian theorem*

For a series satisfying conditions similar to those satisfied by 'slowly increasing' or 'slowly oscillating' functions, an estimate of the order of magnitude of the partial sum can be obtained in case the Riesz partial sum of a given order and type behaves like a prescribed function at infinity. (vide : K. Chandrasekharan and S. Minakshisundaram, *Typical Means*, §1 ·8.) The comparison functions generally considered are non-decreasing. In the present note it is sought to obtain parallel results when the comparison functions may be decreasing.

B. V. Singbal, Bombay : *On the algebraic basis of Morse's variational theory*

Here a new definition of critical levels and critical points is considered and M. Morse's relations are generalized for abelian groups with differential operators with filtration, bringing out thus the algebraic basis of M. Morse's Theory, and for functions on general topological spaces.

T. P. Srinivasan, Bombay : *On extension of measures*

A classical approach due to Caratheodory serves to define a measure on a σ-algebra of sets, starting from an *outer measure*, defined on all sets, axiomatically by its properties. A dual approach through *inner measures*, though desirable, was long considered not workable. It is shown here that with one more assumption on the axiomatic properties defining an inner measure, viz. that it is *regular*, the dual approach does work. It is further shown how the methods of Stone and Daniell to extend linear functionals can be adapted with advantage to extend set functions which are finite measures. Any difficulty in such an adaptation stems from the fact that an "elementary class of functions" has a group theoretic structure, the addition of two functions being the group operation while a ring of sets has none. The first of these methods, introduced here, amounts to metrizing a given class of *measurable* sets and considering its completion with respect to this metric while the second amounts to defining a partial ordering in the given class and taking limits of monotone sequences with respect to this ordering. As each of these methods is shown to be *equivalent* to a third method, described abstractly, it will in particular follow that the two methods are themselves *equivalent*. The advantages of these methods are illustrated by applying them to the construction of regular measures from a *content* in a locally compact Hausdorff space and from a positive linear functional on the class of all continuous functions vanishing outside compact sets. In the course of our discussion, a new proof of the Hahn-Kolmogoroff extension theorem is obtained which incidentally leads to an improvement.

M. S. Narasimhan, Bombay : *The invariance of the first cohomology group of a surface*

H. Weyl has remarked [*Amer. Math. Monthly*, Vol. 57] that his idea of an abstract integral can be used for a direct proof of the invariance of the first Betti number without the tool of the simplicial approximation. Following this remark we give a proof of the invariance of the first cohomology group with real coefficients (defined by means of a triangulation) in the case of a triangulable, orientable surface and also a proof of the invariance of all Betti numbers of a compact orientable surface. All the ideas used in the proof are contained in a recent paper of Weyl [*Zeit. angewandte Math. und Phys.* (*Zamp*), IV, p. 471-492].

V. Venugopal Rao, Bombay : *The zeta function of indefinite quadratic forms*

Let S be a rational, symmetric, non-degenerate, indefinite matrix with signature n, $m-n$. Let A and H be two rational column vectors with m rows. Following C. L. Siegel, consider the equation $S[X+A]=t$; his "measure of representation" is generalized by defining

$$M(S, A, H, t) = \sum_{\substack{X \\ S[Y]=t}} \mu(S, Y)\, e^{2\pi i X'H}$$

the summation being over all non-associated $Y = X + A$ with respect to a certain subgroup of the unit group of S determined by A. It is *necessary* to impose some congruence conditions on A and H to make our definition consistent. Then for complex s,

$$\zeta(S, A, H, s) = \sum_{t>0} \frac{M(S, A, H, t)}{t^s},$$

the summation extending over all positive rational t satisfying $S[X+A]=t$, defines the generalized zeta function. Results analogous to those of Siegel are obtained. (*Math. Zeit.* Bd 44). In addition it is shown that in certain cases $\zeta(S, A, H, s)$ is an *entire* function of s.

By using the functional equation of the Zeta function, the Riesz partial sum $\sum_{t \leqslant x} M(S, A, H, t)(x-t)^a$, $a \geqslant 0$, $x > 0$, is expanded as a series of analytic functions. In the case $|S| > 0$, the expansion involves Bessel functions of the first kind as in the classical case of lattice point problems; on the other hand when $|S| < 0$, Bessel function of the second kind and Bessel functions with imaginary arguments enter.

Generalizations to matrix variables are also being considered.

P. K. Raman, Bombay : *On a class of linear spaces in function theory*

π_r is the space of all power-series $x(\omega) = \sum_{n=1}^{\infty} x_n \omega^{n-1}$ with radius of convergence R, $0 < r \leqslant R \leqslant \infty$; so that π_∞ is the space of entire functions. ρ_r is the space of all power-series $u(z) = \sum_{n=1}^{\infty} u_n z^{n-1}$ with radius of convergence R, $0 \leqslant r < R \leqslant \infty$. π_r and $\rho_{1/r}$ are dual to one another as locally convex spaces.

If $x(\omega), y(\omega) \in \pi_r$ define $x \circ y\,(\omega) = \sum_{n=1}^{\infty} x_n y_n \omega^{n-1}$. This operation is closed in π_r for $r \geqslant 1$. Let $(x)_n(\omega) = x \circ \ldots \circ x(\omega)$ n-times, and $L[x]$ denote the closed linear subspace, in the locally convex topology, generated by $(x)_n(\omega)$, $n = 1, 2, \ldots$. V. G. Iyer has proved that for the space π_∞, $L[x] = \pi_\infty$ when $x(\omega) = \sum_{n=1}^{\infty} x_n \omega^{n-1}$ satisfies the condition (*) (i) $x_i \neq x_j$ for $i \neq j$ and (ii) $|x_i| > 0$. It is proved here that in order that $L[x] = \pi_r$ the condition (*) is necessary for $r \geqslant 1$ and is also sufficient for $r > 1$. A counter-example is given to show that this condition is not sufficient for $r = 1$. Using a known theorem of O. Toeplitz it is shown that when $x(\omega)$ satisfies the condition (*) any $y(\omega) \in \pi_r$ can be approximated uniformly in a circle $|z| \leqslant \dot{\omega} < r$ by means of finite linear combinations of $(x)_n(\omega)$.

C. S. Seshadri, Bombay : *A proof of the functional equations of zeta functions associated with an algebraic number field*

Siegel has given a proof of the functional equation of the zeta function of a totally real algebraic number field, analogous to the first proof of Riemann's, by making use of a generalized Lipschitz formula. It is shown here how a similar proof can be given for the general L-series of Hecke, associated with any algebraic number field by generalizing a formula given by Siegel in his proof of Minkowski's theorem on linear forms. The method developed is also applicable to more general zeta functions.

S. K. Lakshmana Rao, Bangalore : *Gegenbauer transforms*

The Legendre transform of a function $F(x)$ introduced by C. J. Tranter (*Quart. J. of Math.*, Oxford (2), (1950), 1-8) is the sequence of numbers

$$f(n) = \int_{-1}^{+1} F(x)\, P_n(x)\, dx, \quad (n = 0, 1, 2, \ldots),$$

where $P_n(x)$ denotes the Legendre polynomial of degree n. In two recent papers (R. V. Churchill and C. L. Dolph—*Proc. American Math. Soc.* 5 (1954), 93-100., R. V. Churchill, *J. Math. and Phys.* 33 (1954), 165-178) the convolution and other operational properties for the Legendre transform have been developed.

In the present paper the Gegenbauer transform of a function is introduced by the equation

$$f^{\nu}(n) = \int_{-1}^{+1} F(x)\,(1 - x^2)^{\nu - \frac{1}{2}}\, C_n^{\nu}(x)\, dx,\ (\nu > -\tfrac{1}{2}, n = 0, 1, \ldots),$$

where $C_n^{\nu}(x)$ denotes the Gegenbauer polynomial. The convolution theorem and other operational properties of this transform are proved following completely the procedure of the last two papers mentioned.

QAZI IBADUR RAHMAN, Aligarh : *On the coefficients of an entire series of finite order*

Let $f(z) = \sum_{0}^{\infty} a_n z^n$ be an entire function of finite order ρ and let

$$\frac{T}{t} = \overline{\underline{\lim}}_{r\to\infty} \frac{\log M(r)}{r^{\rho}}; \qquad \frac{\phi}{\theta} = \overline{\underline{\lim}}_{r\to\infty} \frac{1}{\rho e}\, n\, |a_n|^{\rho/n}.$$

It is known that $T = \phi$ and $t \geqslant \theta$ with equality sign for the class of functions for which $|a_n/a_{n+1}|$ is a non-decreasing function of n. I take $r^{\rho}(\log r)^{\alpha}$ as comparison function and prove that if

$$\frac{T}{t} = \overline{\underline{\lim}}_{r\to\infty} \frac{\log M(r)}{r^{\rho}(\log r)^{\alpha}}; \quad \frac{\vartheta}{\theta} = \overline{\underline{\lim}}_{r\to\infty} \frac{\rho^{\alpha-1}}{e} \frac{n}{(\log n)^{\alpha}} |a_n|^{\rho/n},$$

then $T = \vartheta$ and $t \geqslant \theta$ with equality sign for the class of functions for which $|a_n/a_{n+1}|$ is a steadily increasing function of n. This result obviously includes the above mentioned results.

M. V. SUBBA RAO, Rajahmundry : *Closure theorems for the space of power series*

S. Mandelbrojt considers " General closure theorems " for the spaces L_p in his Rice Institute Pamphlet of the same title. I have proved similar results here for the space of power series and space of sequences of a certain type. Two results are mentioned below.

Let E be the Banach space of all power series $a(z) = \Sigma a_n z_n$ for which $\Sigma |a_n| R^n < \infty$, ($R$ fixed), the norm of $a(z)$ being the sum of this series. Then

THEOREM I. *Let $a(z) = \Sigma a_n z^n$ be an element of E, $a_n \neq 0$, $n = 1, 2, \ldots$ Let ρ be the radius of convergence of $a(z)$. Let $z_1, z_2, \ldots$ be a sequence of complex numbers such that $|z_n| = < \rho/R$ and having at least one limit point in $|z| = \rho/R$. Let $a_n(z) = a(zz_n)$. Then every element in E can be obtained as a limit in the topology of E, i.e. uniform convergence over the circle $|z| \leqslant R$ of finite linear combinations of the sequence of elements,*

Let S_p be the space of all sequence $n = (a_1, a_2, \ldots,)$ such that $\Sigma |a_n|^p$ is convergent, ($p > 1$ fixed), with norm

$$\|x\| = (\Sigma |a_n|^p)^{1/p}.$$

THEOREM II. *Let $a = (a_1, a_2, \ldots,)$ be an element of S_p, such that $a_n \neq 0, n = 1, 2, \ldots$. Let a_n be the sequence of elements $a_n = (a_1 z_n, a_2 z_n^2, \ldots, a_i z_n^i, \ldots,)$, $n = 1, 2, \ldots$, where (z_n) is a sequence of complex numbers such that $|z_n| \leqslant 1$ and either (i) having a limit point in the unit circle or (ii) $\Sigma (1 - |z_n|)$ divergent. Then the closure of the linear set spanned by a_n is the whole space S_p.*

THEOREM III. *Let $a = (a_1, a_2, \ldots,)$ be an element of S_p such that real part of (a_n) form a strictly increasing sequence. Then the closure of the linear manifold spanned by the elements (a_n) where $a_n = (a_1^n, a_2^n \ldots,)$, $n = 1, 2, \ldots,$ is the whole space S_p.*

T. V. AVADHANI, Waltair : *On the summability of eigenfunction expansions* II

In an earlier paper of mine bearing the same title (I) (*J. Indian Math. Soc.* 18 (1954), pp. 9-18) it was shown that for the class of functions $f(x)$ belonging to L^2, Riesz summability (λ, a) of the Fourier eigenfunction series $\Sigma a_n w_n(x)$ is a local property if $a \geqslant (k-1)/2$. The problem that naturally arises next is that " Given an $a < (k-1)/2$ can we determine the class of functions for which summability is a local property? When is convergence a local property "?

The main object of this paper is to answer the above questions at least in two and three dimensional Euclidean spaces. Incidentally we also establish a slight generalization of Theorem 2 of (I). The basic argument adopted in this paper is essentially the same as in the earlier paper. Theorem 4 is however proved by adopting a slightly modified argument.

THEOREM 1. *If $\Sigma\lambda_n^\nu a_n^2 < \infty$, Riesz summability (λ, a) of the series $\Sigma a_n w_n(x)$ is a local property*

(i) for $a \geqslant (k-1)/2 - \nu$ and zero if $0 \leqslant \nu < 1$, and (ii) for $a > (k-3)/2$ and $a \geqslant 0$ if $\nu = 1$.

THEOREM 2. *In the case of two dimensional Euclidean space, if $f(x) \in Lip\ (\beta, 2)$, $0 < \beta \leqslant \frac{1}{2}$, Riesz summability (λ, a) of the series $\Sigma a_n w_n(x)$ is a local property for $a > \frac{1}{2} - \beta$ whereas for $\beta > \frac{1}{2}$ convergence itself is a local property.*

THEOREM 3. *In the case of three dimensional Euclidean space, if $f(x) \in Lip\ (\beta, 2)$, $0 < \beta \leqslant 1$, Riesz summability (λ, a) of the series $\Sigma a_n u_n(x)$ is a local property if $a > 1 - \beta$,* and

THEOREM 4. *If $\Sigma\lambda_n a_n^2 < \infty$, convergence of the Fourier eigen-function series $\Sigma a_n w_n(x)$ is a local property.*

In all the cases where the summability (convergence) of the series $\Sigma a_n w_n(x)$ is a local property, the necessary additional conditions to be imposed on $f(x)$ in order that the series may be summable (convergent) have also been studied.

S. M. Shah, Aligarh : *Missing values, for simple zeros, of meromorphic functions*

Let $n_1(r, a)$ denote the number of simple zeros of meromorphic function $f(z) - a$, $n_2(r, a)$ the number of simple and double zeros, each counted once only, and $\bar{n}(r, a)$ the number of zeros, each counted once only, in $|z| \leqslant r$. In this paper we prove

(i) *$f(z)$ satisfies the relation*

$$\liminf_{r\to\infty} \frac{T(r)}{n_1(r, x)\log r} \leqslant 10,$$

for every x, with four possible exceptions;

(ii) *$f(z)$ satisfies the relation*

$$\liminf_{r\to\infty} \frac{T(r)}{\bar{n}(r, x)\log r} \leqslant 3,$$

for every x, with two possible exceptions;

(iii) *if for some a_1 (finite or infinite)*

$$\liminf_{r\to\infty} \frac{T(r)}{N_2(r, a_1)} > 24,$$

then

$$\liminf_{r\to\infty} \frac{T(r)}{n_1(r, x)\log r} \leqslant 24,$$

for every x, with three possible exceptions (including a_1).

I prove more sharp results for meromorphic functions of finite positive order and for entire functions. (ii) includes the Picard-Nevanlinna theorem on meromorphic functions.

S. K. Singh, Aligarh : *On exceptional values of entire functions*

Let $f(z)$ be an entire function of order ρ $(0 < \rho < \infty)$, and let S denote the set of all increasing real functions $\psi(x)$ such that $\log x = o(\psi(x))$. We shall call a an *e.v.S.*, if

$$\liminf_{r\to\infty} \frac{\log M(r, f)}{n(r, a)\,\psi(r)} > 0.$$

We prove

Theorem 1. *If a is an e.v.S, then it is an e.v.V (in the sense of Valiron) with the deficiency attained.*

Next we prove

Theorem 2. *If a is an e.v.E, then it is an asymptotic value.*

For the definition of e.v.V, e.v.E and an alternative proof of Theorem 2, see S. M. Shah (Compositio Mathematica, vol. 9, (1951) pp. 227-238).

S. K. Singh, Aligarh : *On the maximum term and the rank of an entire function*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, and let $\mu(r)$ be the maximum term of the series for $|z| = r$ and $\nu(r)$ the rank,

Let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(r)}{r^{\rho}} = \begin{matrix}\alpha\\ \beta\end{matrix}; \quad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\nu(r)}{r^{\rho}} = \begin{matrix}\nu\\ \delta\end{matrix}.$$

S. M. Shah (*Quart. J. Math. Oxford*, 19, (1948), 220-223) has proved that

$$\rho\, \alpha \leqslant \nu \leqslant e\, \rho\, \alpha, \tag{1}$$

$$\delta \leqslant \rho\, \alpha. \tag{2}$$

We prove here that

$$\nu + \delta \leqslant e\, \rho\, \alpha, \tag{3}$$

and that equality cannot hold simultaneously in (2) a

K. PADMAVALLY, Madras : *On the eigenvalues of a symbolic function of a kernel*

Let $f(x) = a_1 x + a_2 x^2 + \ldots + a_n x^n$ be any polynomial in x, and let $f(K) = a_1 K_1 + a_2 K_2 + \ldots + a_n K_n$ where $K(x, y)$ is a given kernel function and K_n the nth iterate of K (K_i being defined by induction as: $K_i(x, y) = \int K(x, z)\, K_{i-1}(z, y)\, dz$, K_1 being $= K$). Then it is well known that if λ is an eigenvalue of K and ϕ the corresponding eigenfunction, $1/f\ (1/\lambda)$ is an eigenvalue of the polynomial kernel $f(K)$ and ϕ a corresponding eigenfunction. T. Sato (*Proc. Physico-Math. Soc.* Japan 23 (1941), 4-7) has proved the converse of this result, namely that: *If λ is an eigenvalue of the polynomial kernel $f(K)$, then at least one root of the equatian $f(1/x) = 1/\lambda$ is an eigenvalue of k.*

The result given by Sato can be extended as follows to the case where $f(x)$ is not necessarily a polynomial :

Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$ be any function of x and $K(x, y)$ a symmetric (this condition is needed only when the range s is infinite) kernel such that

1) $K(x, y)$ is uniformly bounded and integrable in the given range R, which is of the form $x \in S$, $y \in S$;

2) $\int_S |K(x, y)|\, dy$ is $\leqslant$ a constant N, which is less than the radius of convergence of the series $\Sigma a_n x^n$;

3) $\int_R \int |K(x, y)|\, dx\, dy = \int_S dx \int_S |K(x, y)|\, dy = \int_S dy \int_S |K(x, y)|\, dx$ exists.

Then if λ is any eigenvalue of the kernel $f(K) = \sum_{n=1}^{\infty} a_n K_n$, at least one root of the equation $f(1/x) = 1/\lambda$ is an eigenvalue of K.

(The result relating to polynomial kernels has been given in *Methods of Mathematical Physics* by Courant-Hilbert, Ch. III by a method which can be directly extended to any function $f(k)$, for continuous symmetric kernels possessing a discrete spectrum).

This condition is needed only when the range S is infinite.

R. Ballabh, Lucknow: *A note on Poisson's integral formula*

The following form of Poisson's integral formula has been obtained when the value of a harmonic function is given at points of the real axis.

$$f(t_1) = \text{a real constant} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{V_1(t)}{t - t_1}\, dt,$$

t_1 being any point in the upper half t-plane and $V(t)$ the imaginary part of $f(t)$.

This is specially useful in determining the pressure behind a diffracted shock front for linearised pseudo-stationary flows.

APPLIED MATHEMATICS

E. C. George, Bombay: *Turbulence in a mixture of matter and the formation of protogalaxies.*

We have investigated the homogeneous isotropic turbulence in a system consisting of a mixture of electromagnetic radiation and charged matter. Various invariants and covariants are obtained. The problem of gravitational instability and more generally the propagation of fluctuations of energy density and matter density are investigated. It is shown that the analysis succeeds in tracing the evolution of the formation of protogalaxies a stage further than has been achieved so far. Future work is indicated.

J. Ramakanth, Hyderabad: *Finite torsion of a hollow cylinder*

In this paper the problem of finite torsion of a hollow right circular cylinder is solved, when the material has a hexagonal type of aeolotropy. Expressions for stresses displacements have been calculated and twisting movement has been obtained.

Md. Mahfooz Ali Siddiqui, Hyderabad: *Simple tension in finite deformation*

The problem of simple tension is worked out, keeping third order terms in the strain-energy function and using only the two Lamé constants.

The complete solution is discussed starting from a tentative form for the displacements.

A qualitative suggestion is made for the yield stress, in agreement with one given by F. D. Murnaghan. The tension-strain relation is shown to be

$$T = 16\, E\, \sigma \left(S - \frac{1 + 4\sigma}{2\sigma} S^2 + \dots\right).$$

Md. Mahfooz Ali Siddiqui, Hyderabad: *Simple hydrostatic pressure in finite deformation*

The problem of hydrostatic-pressure is also discussed on the same lines.

The following relation between the hydrostatic-pressure radial contraction and bulk modulus is derived:

$$P = \frac{3}{2} K \left[\frac{3}{2} - \frac{4}{(1-S)^2} + \frac{5}{2} \frac{1}{(1-S)^4} \right].$$

B. S. JAIN AND PYARE LAL, Delhi: *Anharmonic pulsations of a two-phase model*

In this paper, the anharmonic oscillations of a two-phase homogeneous model, consisting of a homogeneous core surrounded by an envelope of uniform density, have been studied. The core radius is taken to be half of the total radius of the model and the core density is taken to be 10 times the envelope density for which the period of oscillation for the fundamental mode is about double (in fact 2.18) the period of the first mode. In such a model we except considerable skewness. The problem is solved using numerical method and it is found that for surface amplitude equal to .05, the skewness in the velocity time curve is 1.43 and the increase in the period is 1.2% of the period for the fundamental mode for vanishing amplitude.

GEOMETRY

MRS. KAMALA DEVI SINGH, Delhi: *Infinitesimal deformations of a Riemannian space*

Let V_n be an n-dimensional space with fundamental metric tensor g_{ij}, whose differentiability class is at least two. If we consider an infinitesimal deformation defined by

$$\bar{x}^i = x^i + \epsilon \lambda^i,$$

where ϵ is an infinitesimal constant and λ^i is a vector field. Then the deformed tensor field of any tensor field, say, $T_{ij}{}^h$ is defined by

$$\overline{T}_{ij}{}^h + = T_{ij}{}^h + D\, T_{ij}{}^h$$
$$= T_{ij}{}^h + \epsilon\, (T_{ij,k}{}^h \lambda^k - T_{ij}{}^k \lambda_{,k}{}^h + T_{jk}{}^h \lambda_{,i}{}^k + T_{ik}{}^h \lambda^k{}_{,j}),$$

where D before a tensor field stands for the Lie derivative of that tensor field with respect to the infinitesimal transformation and (,) comma followed by an index denotes covariant derivative with respect to x's. In this paper the infinitesimal deformation of a Riemannian space has been studied and a number of new results have been obtained.

MURLI DHAR UPADHYAY and R. S. MISHRA, Delhi: *Congruences of curves in a Riemannian space*

In this paper we have studied a set of m-n congruences of curves which are such that through each point of a subspace of n dimensions immersed in a Riemannian space of m dimensions a curve of each congruence passed. Focal surfaces, limit surfaces etc. have been defined and studied. The method adopted is an extension of the method used by Weatherburn in the investigations which he made in connection with the congruence of curves in a Euclidean space.

A. R. RAO, Junagadh: *Cyclic perspectivity*

An attempt is made to obtain a configuration of two ordered sets of points, such that one of them is in perspective with all the cyclic permutations of the points of the other.

H. S. NAGARATHNAMMA, Bangalore: *Properties of the second and higher intrinsic derivatives of the unit normal to a Reimannian hypersurface*

If C is a curve in V_n, the components of the intrinsic derivative of the unit normal at any point of C are $q_1{}^a = N_{;i}{}^a \frac{dx^i}{ds}$. We define $q_r{}^a = q^a{}_{r-1;i} \frac{dx^i}{ds}$, $r = 2, 3, \ldots$, as the second, third,..., intrinsic derivatives of the normal. These possess interesting properties. For example, the necessary and sufficient condition that C is of constant "geodesic torsion", is that the second intrinsic derivative of the unit normal is orthogonal to the first binormal vector of the geodesic tangent to C.

K. N. KAMALAMMA, Bangalore: *A note on congruences of Ribaucour*

In this paper it is proved by a straightforward method that the mean ruled surfaces of the congruence of Ribaucour meet the director surface along its lines of curvature.

This result has been proved by R. S. Mishra and Srikrishna in *Ganitha*, 4 (1953).

It is further observed in a congruence of Ribaucour, if the surface of reference and the director surface are inverse surfaces of each other, the congurence and its reciprocal congruence have the same set of mean ruled surfaces.

R. N. KAUL and RAM BEHARI, Delhi: *Generalization of Lie's theorem on null lines*

Lie's theorem on null lines in Euclidean 3-space states that the locus of the middle point of the join of two points, one on the each of two null lines, in space is a minimal surface. It is also known that the locus of a point that divides the join of two points, one on each of two null lines in space, in any constant ratio, is also a minimal surface. The object of this note is to generalise this theorem to a space of n-dimensions.

P. B. BHATTACHARAYA and RAM BEHARI, Delhi: *Congruences of Guichard*

A congruence of Guichard is one in which the focal surfaces are met by the developable surfaces in the lines of curvature. In sections 1 and 2 of this paper we have investigated the conditions under which a system of contravariant vectors $p^a{}_{1/}$ on a surface S_1, determine a congruence of Guichard with the given surface S_1, as one of the focal surfaces. In the particular case when $p^a{}_{1/}$ are the contravariant components of the tangent vector to a one-system of lines of curvature which are parametric on S_1, the general equations have been shown to reduce to known results.

In section 3 we have discussed the spherical representation of a congruence of Guichard incidentally obtaining some known theorems. In section 4 we have shown that when a congruence of Guichard is also a normal congruence, the two focal surfaces become two developable surfaces and that the spherical representations of the developable surfaces through a ray of the congruence are great circles. In the remaining sections we have confined ourselves to the discussion of the five families of ruled surfaces through a ray of a congruence of Guichard and have found interesting properties of the five families.

R. S. MISHRA, Delhi : *A tensor method of finding orthogonal parameters in probability density function*

Let $f(n, x^i)$ be the probability density function of a distribution depending on n parameters x^i $(i = 1,\dots, n)$. The form of the parameters is at our choice and in practical problems it is often convenient to estimate some suitable functions of parameters rather than the parameters themselves. In this paper using the method of tensor calculus I have shown that a p.d.f. of n independent parameters will admit m orthogonal parameters if $m \geqslant \frac{1}{2} n (n+1)$ for all n. The problem has been illustrated by means of examples.

R. S. MISHRA, Delhi : *Generalized Darboux lines in a subspace of a Euclidean space*

Darboux lines in a subspace V_n of a Euclidean space E_m are the curves in V_n which have the property that at each of their points the osculating hyperspheres to the curves touch V_n. Generalizing this concept generalized Darboux lines have been defined as the curves, the osculating hyperspheres of which have the property that the vector joining the centre of the hypersphere to the point P is in the direction of a curve of a given congruence. Various properties of these generalized Darboux lines have been deduced.

R. S. MISHRA, Delhi : *The congruences of curves in a subspace of a non-Riemannian space*

Let a non-Riemannian space V_m be referred to the coordinates y^a $(a = 1,\dots, m)$ and a subvariety V_n be defined by $Q^\sigma (y^1,\dots y^m) = 0$, $(\sigma = n + 1,\dots, m)$, considering a set of $m-n$ congruences of curves in V_m, such that one curve of each congruence passes through each point of the space V_n and putting the vector $\lambda^a{}_{\gamma|}$ in the direction of a curve of the congruence in the form $\lambda_{\tau|}{}^a = t_{\tau|}{}^l \frac{dy^a}{dx^l} + C_\tau{}^\nu \mid v_{\nu|}{}^a$. I have obtained the generalizations of the generalized equations of Gauss and Codazzi in a non-Riemannian space.

A union curve corresponding to a curve $\lambda_{\tau|}$ of a congruence has been defined as a curve in the subspace V_n, such that the vector in the direction of the geodesic curve through a point of the subspace lines in the pencil of directions determined by the tangent vector to the curve C and the first curvature vector of C at P for V_m. The differential equations of these curves have been obtained and studied. Some other curves viz. hyperasymptotic curves etc. have also been defined and new results corresponding to these have been obtained.

K. K. GOROWARA, Delhi : *Ruled surfaces treated with the help of dual coordinates*

Using expressions of the fundamental magnitudes of the first and second order for a ruled surface in terms of dual coordinates various properties of ruled surfaces have been established.

M. K. SINGAL, Delhi : *Central quadric hypersurfaces in a Euclidean space*

If we use Riemannian coordinates y^i with a given point as pole and if a_{ij} be the components of a symmetric covariant tensor of the second order evaluated at the pole, the equation $a_{ij}\, y^i\, y^j = 1$ represents a central quadric hypersurface (or a hyperquadric) with the pole as the centre. In this paper central hyperquadrics have been studied and several theorems relating to hyperplanes and conjugate radii of a central hyperquadric have been obtained.

M. K. SINGAL, Delhi : *Geodesics in a space of positive constant curvature*

Geodesics in a space of positive constant curvature are known to be closed curves of length $2\pi R$, $1/R^2$ being the curvature of the space. In the present paper it has been shown that geodesics in a space of positive constant curvature are curves for which the second curvature vanishes at every point. Regarding a space V_n of positive constant curvature as a hypersphere imbedded in a Euclidean space S_{n+1}, generalizations of several known results in Euclidean space of three dimensions have been obtained.

M. K. SINGAL, Delhi : *Parallel hypersurfaces in a Euclidean space*

If segments of constant length be laid off along normals to a hypersurface of a Euclidean Space, the locus of the other end-points is a hypersurface with the same normals as the given hypersurface. Two hypersurfaces so related are called *parallel hypersurfaces*. In this paper it has been proved that (1) congruences of lines of curvature on a family of parallel hypersurfaces of a Euclidean space correspond; (2) all members of a family of parallel hypersurfaces of a Euclidean space have the same evolute; (3) if among the members of a family of parallel hypersurfaces of a Euclidean space, one member be of positive constant Riemannian curvature, so also are other members of the family.

STATISTICS

M. PERISASTRI, Vizianagram : *A formula for the mode in statistics*

In elementary books on statistics, a formula for the mode x is given by the equation $x = l + c f_2/(f_1 + f_2)$, where l is the lower boundary of the modal class, c the legth of the class interval and f_1, f_2 the frequencies immediately before and immediately after the modal class. I give here a formula for the mode x viz ; $x = x_2 + \frac{1}{2} c \{[(f_2 - f_1) - (f_2 - f_3)]/[(f_2 - f_1) + (f_2 - f_3)]\}$, where x_2 is the mid value of the modal class, c the length of the class interval, f_2 the frequency

in the modal class and f_1, f_2 the frequencies immediately before and immediately after the modal class. The advantage of this formula is that the mode calculated according to this method is nearer to the mean and median than the mode calculated according to the formula given in text books.

A. R. KAMAT, Poona : *A two-sample distribution-free test*

In this paper it is proposed to develop a non-parametric test to test the hypothesis that the two given sets of observations come from populations which have the same spread or dispersion. The two sets of observations are pooled and ranked and the test-criterion proposed is $D_{n,m} = R_n - R_m + m$ where R_n and R_m are the ranges of the ranks assigned to the two sets of observations containing n and m observation respectively. The exact distribution of the statistic $D_{n,m}$ is obtained The distribution is symmetrical when $m = n$. Illustrations are given of the use of the D-test and it is compared with the usual F-test.

A. R. KAMAT, Poona : *Modification mean square successive difference with an exact distribution*

When there is a slow continuous trend in the mean the usual estimator for the variance, viz. the sample variance suffers from a heavy bias. A useful estimator in such circumstances is the mean square successive difference which has been discussed by von Neumann and others in 1941. No exact distribution has been found for this statistic ; hence in this paper a modified form of the same statistic, whose efficiency is comparable to the original one, is proposed and is shown to have an exact distribution. This modified statistic along with another, also having an exact distribution, can be used to set up upper and lower bounds for the probability points of the mean square successive difference. It is also shown that this modified mean square successive difference leads to exact test procedures corresponding to the F-test and the t-test.

TOPOLOGY

V. S. KRISHNAN, Madras : *Asymmetric semi-uniform structures and symmetrization*

The essential part of the axioms of a metric on a space is the triangle axiom, and similarly for the uniform structure on a set the crucial axiom is that each surrounding of the diagonal U contains a $V.V$, where V is another surrounding. Correspondingly in the v-space the chief axiom is the one which says that each neighbourhood U of a point x contains a neighbourhood of each point in a certain neighbourhood V of x. The place of the conditions of symmetry for these types of space structures, a method of associating a symmetric structure to any non-symmetric one, and the relations between these symmetrisations of a semi-metric, semi-uniform, or semi neighbourhood structure is examined. In particular the semi-uniform spaces are characterized in terms of characteristic functions which are upper semi-continuous mappings into the real unit segment. Equivalent characterizations and the derivation of the usual characterizations of uniform spaces are also considered. Some problems, particularly regarding the notion of completeness of the semi-uniform spaces, are indicated.

M. V. Subba Rao, Rajahmundry : *On the lattice product of normed topologies*

Let $E_1 \supset E_2 \supset E_3, \ldots,$ be a sequence of vector spaces and $E = \prod_i E_i$. Let N_i be the norm on E_i, for each i, making E_i complete. Let T be the lattice product topology of the N_i topologies and let D be a dense subset of each E_i under the corresponding N_i topology. Then, it is shown that

(i) E is a complete space in the topology T;

(ii) D is dense in E in the topology T, if the norms are comparable, i.e. for every i and j, of the two topologies defined by $N_i(x)$ and $N_j(x)$ one is stronger than the other.

Using this as a basic result, various applications were considered. Thus, taking the space of power series, $a(z) = \Sigma a_n z^n$, let E_i be the space of power series $a(z)$ with radius of convergence $\geqslant i > 0$; and impose on E_i the norm $N_i\ [a(z)]$ given by $\Sigma_n |a_n| i^n$. Then we have the situation described in the general result. The functionals of the spaces E_i and E (notation same in (1)) are also studied, and we get the interesting result that in this space E, strong and weak convergences are equivalent. It is also shown that E is a non-normable space, and that there are no norms over it topologically weaker than the lattice product topology T of the N_i's.

Applications to other spaces are also considered.

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BOOK REVIEWS

Entire functions, By R. P. Boas, Jr., Academic Press Inc., New York, 1954; pp. 276 + xii; $ 6.

During the last thirty years, a large volume of work has been done in the investigation of the properties of functions of exponential type in an angle and in the whole plane. These results, due to a large body of research workers interested in this field, are found scattered in several journals. The author of the book under review, himself a prominent worker in the field, has done a useful service to those interested in the properties of entire functions, by bringing together in a book all the important results obtained in the field, and giving a systematic account of the results and the main techniques employed in the investigation of these results. The contents of the book, indicated chapterwise below give an idea of the ground covered.

The first chapter collects together the classical results like Jensen's, Carleman's and Nevanlinna's formulae and the Phragmen-Lindelöf theorems. The second chapter gives an account of the classical results on the relation between the order and type of an entire function and the coefficients in its Taylor expansion. This chapter includes the classical results on the relation between the zeros and order of an entire function, order of a canonial product, Hadamard's factorization theorem and the zeros of functions of integral order.

With Chapter three begin what may be called the deeper theorems of the theory of entire functions. This chapter contains all the main results known about the minimum modulus of entire functions. Chapter four deals with the relation between the growth and the density of zeros for functions with real negative zeros (or zeros distributed on a half line).

Chapter five gives a detailed discussion of the indicator function and the indicator diagram of a function of exponential type in an angle or in the whole plane, and may be regarded as an introductory chapter for the investigations in the remaining chapters.

The results in Chapter six centre round the idea that if a function is of exponential type, and its growth is suitably along a line, then its growths along parallel lines are similarly restricted. A large number of results of this type are proved in this chapter. Chapter seven deals with the asymptotic behaviour along half-lines of functions of exponential type in a half plane when restrictions are placed on its growth along the boundary.

Chapter eight discusses the influence of zeros of functions of exponential type on their growth with suitable additional restrictions on the growth of the function along a line.

Chapter nine deals with uniqueness theorems. These are concerned with generalizations of Carlson's theorem that an entire function of exponential type less than π vanishing at the integral points is identically zero. Allied results when the function is assumed to be very small at a sequence of points instead of vanishing actually are also considered.

The main idea in Chapter ten is to derive a relation between the growth of a function of exponential type in a half plane along a line when its growth along a sequence of points on the line is specified along with suitable restrictions on its growth on the boundary of the half plane. This type of problem has attracted the attention of many workers in the field and this chapter contains a systematic account of all the important results in this connection.

If an entire function of exponential type is bounded along the real axis then its derivative is also bounded along the real axis. The generalizations of this theorem form the subject matter of Chapter eleven. Restrictions on the growth of a function lead to similar restrictions on the growth of functions obtained from it by suitably restricted linear operation (e.g. derivation as in the result quoted above).

The twelfth and last chapter contains some applications of the results of the previous chapters to some problems like the complete-

ness of sequence of entire functions, singularities of power series with finite radius of convergence, Fourier series, functions represented by Dirichlet's series and differential equations of infinite order with constant co-efficients. There is an extensive bibliography running over sixteen pages. The printing and get up are attractive.

In a book of this type packed with theorems and their demonstrations, it will be surprising if there are no printing mistakes. Some of the errors could be rectified by a reader interested in the field. I shall content myself with pointing out a few serious errors which actually vitiate the enuciation of the theorem itself and might confuse any one but an expert. On page 4, in the enunciation of Theorem 1.4.2., the over-all hypothesis that $f(z) = O\ (e^{A|z|})$ for a sequence $|z| = r_n \rightarrow \infty$ is omitted. On pages 152-53, the strict inequality $<$ must replace $\leqslant$ in 9.1.2. and 9.1.3. The same error occurs in Theorem 9.1.4. On page 212, in the inequality 11.3.8., $1/\pi$ must be replaced by $\tau\pi$.

V. G.

Advanced trigonometry : By P. Mohan, published by Ram Narain Lal, Allahabad, revised edition (1954), pp. 206, Rs. 3/8.

This book is a revised edition of the author's book first published in 1940 and is intended as a text-book for B.A. and B. Sc. students of North Indian Universities.

This follows the official syllabus closely and contains chapters on complex numbers, Demoivre's theorem, exponential series, hyperbolic functions, logarithms of complex numbers, application to roots of equations, summation of series and infinite products. This is well written and contains a variety of examples and we can recommend this as a text book for college classes. The printing and get up are good.

S. Mahadevan

The Japanese abacus. By Takshi Kojima, Published by Charles E. Tuttle Co., Tokyo (1954) pp. 102, 360 yen or 1 dollar.

In the earliest times the abacus was a reckoning table covered with sand to help to do calculations. This was developed into a ruled table with counters on lines to indicate numbers and later into a frame with beads sliding on rods. This was used throughout Europe ; but after the introduction of Hindu-Arabic decimal notation which simplified calculation, that was discarded. It is now used in Kindergarten classes to help children in counting.

But the abacus used in Japan and China has been developed to a great extent and is now extensively used in all shops and stores for calculation. The latest type consists of an oblong wooden frame 12″ by 2″ holding 21 vertical rods with beads sliding up and down. A beam runs across the board dividing the rods into two sections—the lower with 4 beads and the upper with a single bead.

The book under review explains in detail the use and theory of the abacus. Special exercises are given to develop both the speed and accuracy in the use. The book ends with chapters on decimals, calculations and numerous examples.

This is an interesting book and it is recorded that in a contest, the Japanese abacus scored victories over the electric computer both in speed and accuracy of calculation. This instrument is common in Japan and there are numerous schools to teach boys in its use and there are also abacus research institutes. This abacus, it is claimed, is very cheap and is available with the publishers.

S. Mahadevan

NEWS AND NOTICES

The following persons have been admitted to life-membership in the Society : R. P. Bambah, S. Mahadevan, G. R. Narayana Iyer, M. R. Parameswaran, Alladi Ramakrishnan, G. R. Seth and O. B. Shukla.

The following persons have been elected to membership in Society : Ram Ballabh, Y. G. Godbole, L. C. Gupta, S. Kumar Hindi, Taqdir Husain, B. S. Jain, Sudha Jain, A. R. Kamat, R. N. Kaul, M. K. Kulkarni, D. N. Logany, G. Marburger, R. K. Mishra, V. V. Narlikar, S. J. Oka, Md. Quazi Ibadur Rahaman, M. S. Ramanujan, B. N. Rama Rao, R. Sankar, K. Savithri, S. C. Saxena, N. Sethuraman, C. S. Shah, Hari Shankar, P. L. Sharma, S. S. Shrikhande, J. A. Siddiqi, M. K. Srinivasan, Srirangamma, C. B. L. Varma, P. K. Venkateswaran and R. C. Vyas.

W. T. Guy, Jr., a member of the American Mathematical Society, has been admitted as a member of the Indian Mathematical Society under the reciprocity agreement.

Professor K. Chandrasekharan will be in residence at the Institute for Advanced Study at Princeton, N. J., from May 1, 1955 to October 1, 1955.

Dr. B. N. Prasad of the University of Allahabad attended the Eighth General Conference of UNESCO, from November 12 to December 11, 1954, at Montevideo in Uruguay, South America, as one of the delegates of the Government of India.

A Congress on Theoretical and Applied Mechanics will be held in the month of November, 1955 at the Indian Institute of Technology. Exact dates will be announced later.

The Congress will be on a research level. Papers may be communicated and read on any of the following subjects :—

I. Elasticity—Plasticity—Rheology.

II. Fluid Mechanics— (Aerodynamics—Hydrodynamics).

III. Mechanics of Solids— (Ballistics—Vibrations—Friction—Lubrication).

IV. Statistical Mechanics—Thermodynamics—Heat transfer.

V. Mathematics of Physics and Mechanics—Methods of Computation.

There will be invited addresses of one hour each on special topics. All interested are cordially invited to contribute papers to the Congress.

Members will be given either private hospitality or will be housed in one of the hostels at the Institute.

Registration cards and any other information may be obtained from the Organizing Secretary, Professor B. R. Seth, Indian Institute of Technology, Kharagpur.

International Mathematical Union

The XIth General Assembly of the International Radio Scientific Union (URSI) discussed, and called the attention of the Mathematicians upon, the following problems (communicated by courtesy of Professor B. van der Pol).

1. The propagation of the radio waves through a medium (i.e. troposphere, ionosphere), in which turbulence occurs causing a statistical special distribution of the dielectric constant, which in turn produces a scattering (irregular reflection and diffraction) of the radio waves. (Booker, Carrol, Megaw and others).

2. The relation—if it exists—between the "frequency bandwidth" of a linear electrical circuit and its "time constants" were investigated and discussed. This problem is analogous to the mathematical problem of the geometrical relation between a given function of a real variable and the form of its Fourier cosine and sine transforms.

3. Mean spectrum of a series of impulses which in principle are identical and periodic, but are actually spaced and deformed at random. This problem has been treated by R. Fortet *L' Onde Electrique, 34*, 683, 1954.

If further generalized it can be stated in the following form :

4. *Spectrum of a "random" signal compared with that of a simple signal emitted by the same system.* It is convenient, for determining the spectrum of a given emitter, to apply either single elementary signal $E(t)$ or a periodic sequence of signals $E(t)$. But the spectrum emitted under actual operating conditions, corresponding to a sum,

$$\sum_{i=1}^{n} E(t - t_i)$$

of elementary signals shifted in time, where t_i are random, does not necessarily bear a relation to the spectrum of a single signal or that of a periodical signal.

In all cases of amplitude modulation or pulse modulation a simple solution to the problem of comparing the two types of spectra has been effected easily. In fact, in those cases there corresponds to the sum

$$\sum_{i=1}^{n} E(t - t_i)$$

applied to the input of the system a sum

$$\sum_{i=1}^{n} S(t - t_i)$$

representing the output signal since the system effects only a linear transformation on the signal.

But the problem has yet to be resolved for the case when the transformation is not linear. In particular, in frequency modulation, signal of the form

$$\sum_{i=1}^{n} E(t - t_i).$$

there corresponds an output signal which can be put in complex terms in the form of product

$$\prod_{i=1}^{n} S(t - t_i)$$

and the relationship between the various spectra does not seem to be very simple for those cases when the index of modulation is not very small.

THE RAMANUJAM MEMORIAL PRIZE, 1955

" The Ramanujam Memorial Prize " of the value of Rs. 500 will be awarded for the best essay or thesis written on any branch of Mathematics, embodying the result of the personal investigations of the author and containing clear evidence of independent and original research. The prize is open to all persons born or domiciled in India. Intending competitors should forward their essays or theses so as to reach the Registrar not later than the 1st December 1955.

Note.—The attention of intending competitors for the above prize is specially invited to the following rules and conditions :

1. All essays or theses for the above prize should be sent by registered post, addressed to the Registrar, University of Madras, University Buildings, Chepauk, Madras-5.

2. The essays or theses may be printed, typewritten, lithographed or written, but not in the competitor's own hand.

3. No essay or thesis shall bear upon it the name of the competitor or anything whereby the identity of the author can be

known by the persons to whom the essay or thesis may be sent for examination.

4. Each essay or thesis for the prize shall be headed by the motto or *nom-de-plume*, selected by the candidate and shall be accompanied by a sealed cover bearing on the outside the competitor's own motto or *nom-de-plume* and containing inside, his name, designation and address, and signed declaration that the essay or thesis has been prepared solely for the purpose of competing for the prize and has not formed the basis of a work for which a prize or a degree has been previously awarded.

5. Theses received with open letters will not be accepted.

6. The prize will not be awarded to the same applicant on a second occasion.

THE INDIAN

Statement of Accounts for the year

RECEIPTS

	Rs.	a.	p.	Rs.	a.	p.
By realisation of cheqties pertaining to 1953-54 ...				178	0	0
Subscription from Members and Subscribers :—						
Life Composition Fee	1,025	0	0			
Membership	2,872	13	0			
Subscribers	2,527	10	0			
	6,425	7	0			
Deduct amount paid to American Mathematical Society under reciprocity scheme	48	0	0			
				6,377	7	0
By Sale of Publications				1,069	15	0
Grants-in-aid :						
Tata Institute of Fundamental Research ...	1,000	0	0			
National Institute of Sciences	1,000	0	0			
Madras University	150	0	0			
Osmania University	100	0	0			
				2,250	0	0
Interest :						
Fixed Deposit	840	0	0			
Current Account	14	9	0			
Savings Account	14	11	9			
				869	4	9
Total receipts during 1954-55				10,744	10	9
Amount transferred from F. D. to C/Acct. ...				4,000	0	0
Opening Balance :						
Current Account at Indian Bank, Ltd., Madras ...	4,548	10	6			
Current Account at Bank of Mysore, Ltd., Bangalore	3,870	9	9			
				8,419	4	3
Savings Account at Indian Bank, Madras				456	3	4
Fixed Deposit	16,000	0	0			
Less : Transferred to Current Account	4,000	0	0			
				12,000	0	0
Dr. A. Narasinga Rao Medal Fund (in the form of National Savings Certificates)				1,000	0	0
National Savings Certificates (Face Value)				9,000	0	0
Cash on Hand with Dr. C. N. Srinivasiengar ...				2,015	0	0
Cash on Hand with Dr. K. Chandrasekharan ...				219	4	6
Total Rs.				47,854	6	10
Deduct amount used for Dr. Narasinga Rao's Medal				96	6	0
Total Rs.				47,758	0	10

Checked and found correct.

B. Shamanna,

Chartered Accountant.

21*st April* 1955.

MATHEMATICAL SOCIETY

April 1954 to March 1955

EXPENDITURE

	Rs.	a.	p.	Rs.	a.	p.
Printing				8,360	3	0
Office Expenses				891	10	6
Library				3,646	15	6
Dr. A. Narasinga Rao Gold Medal				96	6	0
Other Expenses				71	6	0
Total expenses during 1954-55				13,066	9	0
Closing Balance :						
Current Account at Indian Bank, Madras	5,315	12	3			
Current Account at Bank of Mysore, Bangalore ...	6,093	13	3			
				11,409	9	6
Fixed deposit				12,000	0	0
Savings Account				890	15	1
Dr. A. Narasinga Rao Gold Medal Fund (in the form of National Savings Certificates less amount utilised)				903	10	0
National Savings Certificates (General Account Face Value)				9,000	0	0
Advance with Dr. Chandrasekharan				487	5	3
Total Rs.				47,758	0	10

C. N. SRINIVASIENGAR,
Hon. Treasurer.

THE INDIAN

Balance Sheet for the year

CAPITAL AND LIABILITY

		Rs. a. p.	Rs. a. p.
CAPITAL—			
Opening Balance			37,287 12 1
Less this year's excess expenditure as shown below:—			
Total expenditure as per expenditure statement ...		13,066 9 0	
Less : Total receipts as per receipt statement ...		10,744 10 9	
		2,321 14 3	
Add : Dr. A. Narasinga Rao Gold Medal	96 6 0		
Add : Last year's cheque realisation ...	178 0 0		
		274 6 0	
			2,596 4 3
Total Rs.			34,691 7 10

This is to certify that the receipt and expenditure statements for the period
and verified with the receipt books, Bank Pass Books, Money order coupons and
in my opinion the above Balance sheet exhibits a true and correct view of the state
the best of my information and explanation given to me and as shown by the
taken in getting the Library and Furniture of the Society valued and certified.
to an audit. If the cost of Books is not shown, the Balance Sheet will not show

Malleswaram, Bangalore.

B. SHAMANNA, F.C.A.,
Chartered Accountant.
21st April 1955.

MATHEMATICAL SOCIETY

April 1954 to March 1955

ASSETS AND PROPERTY

	Rs.	a.	p.	Rs.	a.	p
Current Account at Indian Bank, Ltd., Madras ...	5,315	12	3			
Current Account at Bank of Mysore, Ltd., Bangalore	6,093	13	3			
				11,409	9	6
Fixed Deposit				12,000	0	0
Savings Bank Account				890	15	1
Dr. A. Narasinga Rao Gold Medal Fund (in the form of National Savings Certificate)	1,000	0	0			
	96	6	0			
				903	10	0
National Savings Certificate (General account) Face Value				9,000	0	0
Advance with Dr. K. Chandrasekharan				487	5	3
Total Rs.				34,691	7	10

ending 31st March 1955, of the Indian Mathematical Society have been checked vouchers and found correct. The Balance sheet has also been examined by me and of affairs of the Indian Mathematical Society as on March 31st 1955 according to Books of the Society. I have to make the observation that no steps have been A duly authorised certificate as regards stock verification is a necessary adjunct the true state of affairs.

C. N. SRINIVASIENGAR,
Hon. Treasurer.

A COMBINATORIAL PROOF OF AN INEQUALITY DUE TO HUA

By L. C. HSU

The following theorem, as stated in a modified form, was obtained a few years ago by Professor L. K. Hua :

Let $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k$ *be two sets of points in a metric space in which the distance function is* $\rho(x, y)$. *Let* T *be any positive constant. Then the following conditions*

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \rho(\alpha_i, \alpha_j) \leqslant T, \ \sum_{i=1}^{k}\sum_{j=1}^{k} \rho(\beta_i, \beta_j) \leqslant T, \ \rho(\alpha_i, \alpha_j) + \rho(\beta_i, \beta_j) > 0,$$

imply the inequality

$$\min\Big(\rho(\alpha_i, \alpha_j) > 0, \ \rho(\beta_i, \beta_j) > 0\Big) \leqslant \frac{3T}{2k\,(k-1)}. \tag{1}$$

The conclusion states that the minimum among all those distances $\rho(\alpha_i, \alpha_j)$'s, $\rho(\beta_i, \beta_j)$'s which are not zero, does not exceed $\dfrac{3\,T}{2k\,(k-1)}$. The proposition is obvious for $k \leqslant 3$. In what follows we shall assume $k \geqslant 3$ and give a combinatorial proof of the result.

Let A, B be the numbers of zero terms contained in $\sum_{i<j} \rho(\alpha_i, \alpha_j)$, $\sum_{i<j} \rho(\beta_i, \beta_j)$ respectively, and let

$$\alpha_1 = \ldots = \alpha_{s_1}; \ldots ; \alpha_{s_{r-1}+1} = \ldots = \alpha_{s_{r-1}+s_r},$$

where $1 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_r$, $s_1 + s_2 + \ldots + s_r = k$. Then we have

$$A = A^{(r)} = \binom{s_1}{2} + \binom{s_2}{2} + \cdots + \binom{s_r}{2}. \tag{2}$$

In order to determine the maximum value of B, taking account of the condition $\rho(\alpha_i, \alpha_j) + \rho(\beta_i. \beta_i) > 0$, and the fact that

$$\binom{m+n}{2} \geqslant \binom{m}{2} + \binom{n}{2},$$

we may assume

$$\beta_1 = \beta_{s_1+1} = \cdots = \beta_{s_{r-1}+1}; \ \beta_2 = \beta_{s_1+2} = \cdots = \beta_{s_{r-1}+2}; \cdots$$

Thus the maximum of B, denoted by $B_{\max}$, is easily found to be

$$B_{\max} = B^{(r)}_{\max} = s_1 \binom{r}{2} + (s_2 - s_1) \binom{r-1}{2} + \ldots + (s_{r-1} - s_{r-2}) \binom{2}{2}$$
$$= s_1 (r-1) + s_2 (r-2) + \ldots + s_{r-1}. 1 \tag{3}$$

Now it suffices to show that

$$\min (A^{(r)}, B^{(r)}_{\max}) \leqslant \frac{1}{3} \binom{k}{2} = \frac{k(k-1)}{6}. \tag{4}$$

For, when this is done, we shall see that one of the sums

$$\sum_{i<j} \rho(\alpha_i, \alpha_j) \ (\leqslant \tfrac{1}{2} T), \ \sum_{i<j} \rho(\beta_i, \beta_j) \ (\leqslant \tfrac{1}{2} T)$$

will contain at least

$$\binom{k}{2} - \frac{1}{3}\binom{k}{2} = \frac{2}{3}\binom{k}{2}$$

non-zero terms so that (1) will be implied by

$$\min (\rho(\alpha_i, \alpha_j) > 0, \rho(\beta_i, \beta_j) > 0) \leqslant \frac{T}{2} \left\{ \frac{2}{3} \binom{k}{2} \right\} = \frac{3T}{2k(k-1)}. \tag{5}$$

The following procedure is merely a justification of (4) by induction on r. Replace r by ν and write

$$k_\nu = s_1 + s_2 + \ldots + s_\nu, \ 1 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_\nu, \ (\nu = 1, 2, 3, \ldots).$$

Evidently (4) is true for $r = 1$, since $B^{(1)}_{\max} = s_1 (1 - 1) = 0$. We now suppose (4) is true for $r = \nu$ and prove its truth for $r = \nu + 1$. Our supposition gives

$$\min (A^{(\nu)}, B^{(\nu)}_{\max}) \leqslant \frac{k_\nu (k_\nu - 1)}{6}. \tag{6}$$

(i) If $A^{(\nu)} \leqslant \frac{1}{6} k_\nu (k_\nu - 1)$, $s_\nu \leqslant s_{\nu+1} \leqslant k_\nu$, then

$$A^{(\nu+1)} = A^{(\nu)} + \binom{s_{\nu+1}}{2}$$
$$= A^{(\nu)} + \frac{s_{\nu+1} (s_{\nu+1} - 1)}{2}$$
$$< \frac{k_\nu (k_\nu - 1)}{6} + \frac{s_{\nu+1} (2k_\nu + s_{\nu+1} - 1)}{6}$$
$$= \frac{(k_\nu + s_{\nu+1})(k_\nu + s_{\nu+1} - 1)}{6} = \frac{1}{3}\binom{k_{\nu+1}}{2},$$

where $k_{\nu+1} = k_\nu + s_{\nu+1} = s_1 + \ldots + s_\nu + s_{\nu+1}$.

(ii) If $B^{(\nu)}_{\max} \leqslant \frac{1}{6} k_\nu (k_\nu - 1)$, $3 \leqslant s_{\nu+1}$, then

$$B^{(\nu+1)}_{\max} = s_1 \nu + s_2 (\nu - 1) + \ldots + s_\nu . 1$$
$$= s_1 (\nu - 1) + \ldots + s_{\nu-1} . 1 + (s_1 + s_2 + \ldots + s_\nu)$$
$$= B^{(\nu)}_{\max} + k_\nu \leqslant \frac{k_\nu (k_\nu - 1)}{6} + \frac{s_{\nu+1}(2k_\nu + s_{\nu+1} - 1)}{6} = \frac{1}{3}\binom{k_{\nu+1}}{2}.$$

The cases $3 > s_{\nu+1}$ in (ii) and $s_{\nu+1} > k_\nu$ in (i) remain to be discussed.

(iii) Supposing $s_1 = \ldots = s_t = 1$, $s_{t+1} = \ldots = s_\nu = 2$ $(0 \leqslant t \leqslant \nu)$ and $k_\nu \geqslant 3$, we get, by means of (2),

$$A^{(\nu)} = \sum_1^{\nu - t} \binom{2}{2} = \nu - t \leqslant \frac{1}{3}\binom{2(\nu - t) + t}{2}$$
$$= \frac{1}{3}\binom{s_1 + \ldots + s_\nu}{2} = \frac{1}{3}\binom{k_\nu}{2}.$$

This is obviously valid for all $\nu = 2, 3, \ldots$, provided $s_\nu < 3$.

(iv) Supposing $1 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_\nu \leqslant s_{\nu+1}$, $3 < k_\nu = s_1 + \ldots + s_\nu < s_{\nu+1}$, we obtain

$$k_{\nu+1} = k_\nu + s_{\nu+1} > 2k_\nu, \text{ or } k_{\nu+1} \geqslant 2k_\nu + 1,$$
$$B^{(\nu+1)}_{\max} = s_1 \nu + s_2 (\nu - 1) + \ldots + s_\nu . 1$$
$$\leqslant \frac{s_1 + s_2 + \ldots + s_\nu}{\nu} + (\nu + (\nu - 1) + \ldots + 1)$$
$$= \tfrac{1}{2} k_\nu (\nu + 1).$$

Hence

$$\frac{1}{3}\binom{k_{\nu+1}}{2} = \frac{k_{\nu+1}(k_{\nu+1} - 1)}{6} \geqslant \frac{(2k_\nu + 1)(2k_\nu)}{6}$$
$$> \tfrac{2}{3} k_\nu^2 \geqslant \tfrac{1}{2} k_\nu (\nu + 1) \geqslant B^{(\nu+1)}_{\max}.$$

So we have proved for each case that $\min (A^{(\nu)}, B^{(\nu)}_{\max}) \leqslant \frac{1}{3}\binom{k_\nu}{2}$ implies $\min (A^{(\nu+1)}, B^{(\nu+1)}_{\max}) \leqslant \frac{1}{3}\binom{k_{\nu+1}}{2}$.

Our induction process is therefore completed.

In the foregoing proof, it is obvious that the triangle inequality for the distance function of a metric space,

$$\rho(x, y) + \rho(y, z) \geqslant \rho(x, z),$$

has never been used. Therefore, it may be remarked that the theorem just proved does also hold for the case when $\rho(x, y)$ is replaced by any real valued function $f(x, y)$ defined for every pair of elements of a metric space such that

$$f(x, y) \begin{cases} = 0 & \text{for} \quad x = y. \\ > 0 & \text{for} \quad x \neq y, \end{cases}$$

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ON THE EQUATION $X^3 + Y^3 = U^3 + V^3$

By P. KESAVA MENON

Let k be the field of rational numbers and $\rho = \frac{1}{2}(-1 + \sqrt{-3})$ a cube root of unity. Let $K = k(\rho)$. Every integer of K can be uniquely written in the form

$$p + \rho q,$$

where p, q are rational integers. Also in K, unique factorization into product of primes holds good.

Let x, y, u, v be a rational integral solution of

$$X^3 + Y^3 = U^3 + V^3. \tag{1}$$

It is said to be *trivial* if $x = -y$, $u = -v$ or $x = u$ or v and $y = v$ or u. A nontrivial solution is said to be *primitive* if

$$(x, y, u, v) = 1. \tag{2}$$

Every solution is a multiple of a primitive solution. We shall be concerned with primitive solutions.

Let x, y, u, v be a primitive solution of (1). Let $p + \rho q$, $r + \rho s$ be two integers of K satisfying

$$(x + \rho y)(p + \rho q) = (u + \rho v)(r + \rho s). \tag{3}$$

Taking norms we have

$$(x^2 - xy + y^2)(p^2 - pq + q^2) = (u^2 - uv + v^2)(r^2 - rs + s^2). \tag{4}$$

But from (1) we have

$$(x + y)(x^2 - xy + y^2) = (u + v)(u^2 - uv + v^2),$$

so that

$$(x + y)(r^2 - rs + s^2) = (u + v)(p^2 - pq + q^2). \tag{5}$$

Multiply both sides of (3) by $p + \rho^2 q$ and equate real and imaginary parts; we get

$$(p^2 - pq + q^2)\, x = (pr - qr + qs)\, u + (qr - ps)\, v, \tag{6}$$

$$(p^2 - pq + q^2) y = (ps - qr)\, u + (pr + qs - ps)\, v. \tag{7}$$

Using (5) and the abbreviations

$$\left.\begin{aligned} P &= p^2 - pq + q^2,\ R = (p+q)\,(r+s) - qr \\ Q &= r^2 - rs + s^2,\ S = (p+q)\,(r+s) - ps \end{aligned}\right\}, \tag{8}$$

we get

$$(P^2 - QR)\, u = (QS - P^2)\, v. \tag{9}$$

Let $\mu = (u, v)$. Then for an integer λ

$$\left.\begin{aligned} \frac{\lambda}{\mu}\, u &= QS - P^2 \\ \frac{\lambda}{\mu}\, v &= P^2 - QR \end{aligned}\right\}. \tag{10}$$

Furthermore because of (2), $(\mu, x) = (\mu, y) = 1$ so that $\mu | \lambda$. Hence using (6) and (7) we get

$$\left.\begin{aligned} \nu x &= Q^2 - PR,\ \nu y = PS - Q^2 \\ \nu u &= QS - P^2,\ \nu v = P^2 - QR \end{aligned}\right\} \tag{11}$$

Since $(x, y, u, v) = 1$, it follows that

$$\nu = (Q^2 - PR,\ QS - P^2,\ PS - Q^2,\ P^2 - QR). \tag{12}$$

If x, y, u, v is not a primitive solution, let $t = (x, y, u, v)$; then $x/t,\ y/t,\ u/t,\ v/t$ is a primitive solution and there exist integers p, q, r, s, ν satisfying (11). Conversely if p, q, r, s are any integers then the x, y, u, v satisfying (11) and (12) are solutions of (1) and constitute a primitive solution. Since P, Q, R, S are polynomials in p, q, r, s, it is always possible to choose p, q, r, s in such a manner that the solution obtained is not trivial. Hence

THEOREM. *Every primitive solution of* (1) *corresponds to a set of integers p, q, r, s satisfying* (8), (11) *and* (12), *and to every set of integers p, q, r, s, the integers x, y, u, v defined by* (11) *and* (12) *form a primitive solution of* (1).

Let us put $p_0 + \rho q_0 = \dfrac{p + \rho q}{r + \rho s}$ and let x, y, u, v be a primitive solution. Then

$$(x + \rho y)(p_0 + \rho q_0) = (u + \rho v). \tag{13}$$

Let $p_0 + \rho q_0 = a + b\sqrt{-3}$; p_0, q_0, a, b are rational numbers. Then

$$p_0 = a + b, \quad q_0 = 2b, \tag{14}$$

and

$$a = p_0 - q_0/2, \; b = q_0/2. \tag{15}$$

Let P_0, Q_0, R_0, S_0 be the values of P, Q, R, S. Then

$$\left.\begin{aligned} P_0 &= a^2 + 3b^2, & R_0 &= a - 3b, \\ Q_0 &= 1, & S_0 &= a + 3b, \end{aligned}\right\} \tag{16}$$

and so we get

$$\left.\begin{aligned} \nu x &= 1 - (a^2 + 3b^2)(a - 3b), & \nu u &= (a + 3b) - (a^2 + 3b^2)^2 \\ \nu y &= (a + 3b)(a^2 + 3b^2) - 1, & \nu v &= (a^2 + 3b^2)^2 - (a - 3b). \end{aligned}\right\}. \tag{17}$$

So any primitive solution gives a value of p, q, r, s, ν and so of p_0, q_0 and hence of a, b, ν. Also (14) shows that if a', b' are two other rational numbers to which corresponds a primitive solution x', y', u', v' then by (13), x, y, u, v and x', y', u', v' are distinct. Hence

THEOREM. *To every primitive solution of* (1) *there exists one and only one set of values of a and b which are determined by* (17) *for appropriate rational* ν.

REFERENCE

HARDY and WRIGHT: *Introduction to the theory of numbers*, Oxford, 1940.

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SOME PROPERTIES OF QUADRATIC RESIDUES

By HANSRAJ GUPTA

INTRODUCTION. Let p be an odd prime and μ any positive integer. We denote by R the set of quadratic residues modulo p^μ and by N the set of quadratic non-residues modulo p^μ. We denote by r an arbitrary element of R and by n an arbitrary element of N. The number of elements in each of the sets R and N is $\frac{1}{2}\phi(p^\mu)$ and the elements are all prime to p.

If S be any set of integers and t an arbitrary integer, then we indicate by $t \oplus S$ the set of integers obtained by adding t to each of the elements of S. We denote by i and e arbitrary odd and even integers.

j denotes $p^{\mu-1}\,[(p/4)]$ and k denotes $p^{\mu-1}\,[(p-3)/4]$, where $[x]$ denotes as usual the greatest integer in x.

$m = 0$ or $p^{\mu-1}$ according as $p \equiv 1$ or $-1 \pmod 4$.

In this paper, we find the number of quadratic residues and quadratic non-residues modulo p^μ in the sets $n \oplus R$, $r \oplus N$, $n \oplus N$ and $r \oplus R$. The results proved are generalizations of two well-known theorems recently stated and proved by Kelly [1]. Reference may also be made to the work of Perron [2] cited by Kelly. The paper has, however, not been accessible to me.

THEOREM 1. *Each of the sets $n \oplus R$ and $r \oplus N$ has j quadratic residues and j quadratic non-residues modulo p^μ besides m multiples of p.*

PROOF. First, let $p \equiv 1 \pmod 4$. If g be a primitive root of p^μ, then $n = g^i$ and $r = g^e$. Keeping i fixed, let e run through the $\frac{1}{2}\phi(p^\mu) = 2q$ consecutive even values

$$i-2q+1,\ i-2q+3,\ i-2q+5, \ldots,\ i+2q-1.$$

Then the elements of the set $n \oplus R$ are

$$g^{i-2q+1}(1+g^{2q-1}),\ g^{i-2q+3}(1+g^{2q-3}),\ldots,g^{i-1}(1+g),\ g^{i}(1+g),\ \ldots,g^{i}(1+g^{2q-1}).$$

Since p is of the form $4t+1$,

$$1+g^{s}\not\equiv 0 \pmod p$$

for any odd s. The members of the set $n \oplus R$ are, therefore, all prime to p and incongruent modulo p^{μ}. Moreover, of the numbers equidistant from the beginning and the end in the set $n \oplus R$ one is a quadratic residue and the other a quadratic non-residue modulo p^{μ}. Hence the quadratic residues and non-residues in $n \oplus R$ are equinumerous and their number is $p^{\mu-1}(p-1)/4$, i.e. j.

Secondly, let $p \equiv -1 \pmod 4$. Then

$$1+g^{t}\equiv 0 \pmod p,$$

if t is an odd multiple of $\dfrac{p-1}{2}$. Thus there are $p^{\mu-1}$ multiples of p among the members of $n \oplus R$. The remaining

$$\tfrac{1}{2}\,\phi(p^{\mu})-p^{\mu-1}=\tfrac{1}{2}p^{\mu-1}(p-3)$$

elements of $n \oplus R$ can be paired off as before into quadratic residues and non-residues modulo p^{μ}. Hence the number of quadratic residues and non-residues is given by $\frac{1}{4}p^{\mu-1}(p-3)$, i.e. j.

The case of the set $r \oplus N$ can be dealt with in the same way.

Theorem 2. *The set $r \oplus R$ has k quadratic residues and $k+p^{\mu-1}$ non-residues modulo p^{μ}, while the set $n \oplus N$ has $k+p^{\mu-1}$ quadratic residues and k non-residues modulo p^{μ}.*

Let S denote the set consisting of the elements of both N and R. Then $r \oplus S$ consists of the elements

$$g^{e}(1+g),\ g^{e}(1+g^{2}),\ g^{e}(1+g^{3}),\ldots,\ g^{e}(1+g^{\phi(p^{\mu})}).$$

Now

$$1+g^{t}\equiv 0 \pmod p$$

if t is an odd multiple of $\dfrac{p-1}{2}$. Thus among the elements of $r \oplus S$, there are exactly $p^{\mu-1}$ multiples of p. This means a loss of $p^{\mu-1}$ to the set of quadratic residues because the numbers $1, 1+p$,

$1+2p$, etc. belonging to the quadratic residue set do not occur in the set $1+g^t$, $1 \leqslant t \leqslant \phi(p^\mu)$, and the numbers $1+g^t$, $1 \leqslant t \leqslant \phi(p^\mu)$ are incongruent modulo p^μ. Hence the set $r \oplus S$ has $p^{\mu-1}$ fewer quadratic residues than non-residues modulo p^μ. The number of quadratic residues modulo p^μ in $r \oplus S$ is, therefore,

$$\tfrac{1}{2}\phi(p^\mu) - p^{\mu-1}.$$

Of these j quadratic residues belong to $r \oplus N$. Hence the number of quadratic residues in $r \oplus R$ is

$$\tfrac{1}{2}\phi(p^\mu) - p^{\mu-1} - p^{\mu-1}\,[(p/4)], \text{ i.e. } k;$$

while the number of quadratic non-residues is given by

$$\tfrac{1}{2}\phi(p^\mu) - p^{\mu-1}\,[(p/4)], \text{ i.e. } k + p^{\mu-1}.$$

The second part of the theorem can be proved in the same way.

REFERENCES

1. J. B. Kelly: A characteristic property of quadratic residues, *Proc. American Math. Soc.* 5 (1954), 38-46

2. O. Perron: Bemerkungen über die Verteilung der quadratische Reste, *Math. Zeit.* 56 (1952), 122-130.

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CLASSROOM NOTES

A note on Newton's method for evaluating roots of $f(x) = 0$

By D. K. SEN, *Bombay*

In an earlier paper [*Mathematics Student*, Vol. XIX, Nos. 1 & 2, 1951] I had shown that Newton's method can be successfully applied to the case of nearly equal roots of $f(x) = 0$. Mr. Sohon [*Engineering Mathematics*] has given some interesting diagrams illustrating the use of Newton's method with some remarks of a general inconclusive nature. Whittaker and Robinson [*Calculus of Observations*] and Runge and König [*Numerischen Rechnen*] have also made some useful and definite statements on this method. I have studied the applicability of Newton's formula to various cases, and have been able to formulate some useful new rules, viz. (3), (4), (5) and (5*a*) of the present paper.

The geometrical method of deducing Newton's formula has the advantage over the one based on Taylor's theorem, in that it has not to assume that the terms $\frac{1}{2}f''(x)h^2, \ldots$ are small in comparison with $[f(x) + hf'(x)]$. The only criterion of success of Newton's formula obtained by the geometrical process is that a sequence of points should converge—a criterion which is easily seen to be satisfied or not from the nature of the graph of $y = f(x)$ near the root. I have therefore used the geometrical approach in my treatment of the problem.

Newton's formula is

$$x_{r+1} = x_r - f(x_r)/f'(x_r), \tag{1}$$

where x_r and x_{r+1} are two successive approximate values of the root. From this it follows that

$x_{r+1} <$ or $> x_r$, according as $f(x_r)$ and $f'(x_r)$ have like or unlike signs. (2)

Let neither $f'(x)$ nor $f''(x)$ be zero or infinite at the root $(x = c)$. It is always possible in such a case to obtain an interval including the root in which $f'(x)$ *and* $f''(x)$ *preserve their respective signs.* One can then easily get the following rules from geometrical considerations of concavity or convexity :

$$x_{r+1} < c \text{ when } f'(x) \text{ and } f''(x) \text{ have opposite signs ;} \tag{3}$$

$$x_{r+1} \geqslant c, \text{ when } f'(x) \text{ and } f''(x) \text{ have the same sign,} \tag{4}$$

irrespective of whether $x_r <$ *or* $> c$.

Further, when $f'(x)$ and $f''(x)$ have opposite signs and $x_r < c$, or when $f'(x)$ and $f''(x)$ have the same sign and $x_r > c$,

x_{r+1} *lies on the same side of c as* x_r *and is nearer to c than* x_r. (5)

Otherwise x_{r+1} and x_r will be on opposite sides of c. But even in this case the points $x_{r+1}, x_{r+2}, \ldots,$ are on same side of c and converge to c. (5*a*)

From the consideration of concavity or convexity towards the foot of the ordinate we get the following rules which are not so useful as the ones given above.

If $f'(x)$ and $f''(x)$ be not zero between x_r and the root c and if the product $f(x_r) \,.\, f''(x_r)$ is positive, then x_{r+1} remains on the same side of c and is closer to c than x_r; but if the above product is negative, then x_{r+1} crosses over to the other side of c. (6)

If $f'(x)$ is zero once between x_r and the root, $f''(x)$ be not zero, and if the above product is negative, then x_{r+1} moves further away from c than x_r. (7)

Another useful rule [D. K. Sen, *Mathematics Student*, Vol. XIX, Nos. 1 & 2, 1951] is that if $f'(x) = 0$ has a root p lying between two roots of $f(x) = 0$, then Newton's formula will lead to the smaller or the greater of the two roots according as x is selected $<$ or $> p$, provided that $f'(x) = 0$ has no other root lying between x and p. (8)

A judicious choice of the value to be used in Newton's formula at any stage, for the determination of the next value, can considerably

shorten the calculation for the final result. It is not necessary to use the value x_r obtained at any stage, and it will be found that a slightly different value, properly chosen, will produce a quicker and better result. This can be done with the help of the rules (3) and (4).

ILLUSTRATION. The equation

$$f(x) \equiv x^4 + 4x^3 - 6x^2 + 2.625x - 0.37158 = 0, \qquad (A)$$

has three positive roots close to each other and the fourth root is negative. Both $f'(x)$ and $f''(x)$ vanish and are small in the neighbourhood of the positive roots of $f(x) = 0$. With the help of the rules, however, it is possible to obtain all the roots without much difficulty.

(i) *Negative root.* It can be easily found that $f(x)$ changes sign from + to − in the interval (− 5.3, − 5.2), and that $f'(x)$ and $f''(x)$ have opposite signs in this interval. Using $x_1 = 5.3$ (< root), we get $x_2 = -5.244$, which is also < root, by Rule (3). Hence we use $x_2' = -5.24$ and obtain $x_3 = -5.24257$ (still < root). Using next $x_3' = -5.2425$, we get in a similar manner, $x_4 = -5.242565$.

(ii) *Second root.* We find that $f(0.3)$ is negative, $f(0.4)$ is positive, both being quite small, and that $f'(x)$ and $f''(x)$ have opposite signs here. We therefore proceed as in the case of the negative root, with $x_1 = 0.4$, and obtain $x_4 = 0.393131$.

(iii) *Other roots.* We note that $f''(x)$ changes sign in the interval (0.4, 0.5), $f(x)$ is quite small, but $f'(0.4)$ and $f'(0.5)$ are positive and small. As trials, $x_1 = 0.41$ leads to a greater value and $x_1 = 0.5$ leads to a smaller value for x_2. These indicate that $f(x)$ vanishes in this interval and that there are two roots of $f(x) = 0$ here. We therefore use smaller intervals and obtain the following :

$f(0.41)$ is +; $f(0.42)$ is −; $f(0.43)$ is −; $f(0.5)$ is +;

$f'(0.41)$ is −; $f'(0.42)$ is −; $f'(x)$ continues to be +;

$f''(0.41)$ is −; $f''(0.42)$ is +; $f''(x)$ continues to be +;

since $f'(x)$ and $f''(x)$ preserve their signs in the interval (0.43,0.5) wherein lies a root. Taking $x = 0.43$ we get $x_4 = 0.4344539$. The last root lies in (0.41,0.42) where $f''(x)$ changes sign. Hence we

have to take still smaller intervals. Linear interpolations show that 0.41499 is closer to the root of $f(x) = 0$, than 0.41 ; and 0.41420 is a closer approach to the root of $f''(x)=0$.

We take a number, say 0.4144, between these values and see that $f(x)$ is $+$, $f'(x)$ is $-$ and $f''(x)$ is $+$ at $x = 0.4144$. Hence the root lies in the interval (0.4144, 0.42) and $f'(x)$ and $f''(x)$ preserve their signs. We put $x_1 = 0.4144$ and get $x_2 = 0.4149786$, which is quite close to, but less than the required root.

Suppose that we had observed that

$$f(0.30) = -0.00798 \text{ and } f(0.41) = 0.00001.$$

To find a better value of the root we would naturally use $x_1 = 0.41$ in Newton's formula ; contrary to our expectations we would find that $x_2 = 0.415$, a value outside the interval (0.30,0.41). And if we persist in using this value of x_2 for the next approximation, we will find that the new values too remain beyond 0.41. The real reason for this behaviour of the successive values is that $f'(x)$ changes sign in this interval; hence we have to take smaller intervals like (0.30,0.40) to get the desired root (cf. the case of second root).

When $f'(x)$ or $f''(x)$ is zero or infinite *at the root of* $f(x) = 0$, the root can be found by more direct methods, e.g. the method of H.C.F., than by Newton's formula. If, however, one so desires, Newton's formula can also be used with suitable precautions. Thus, when $f'(x) = 0$ at the root, x_1 is to be so selected that $f''(x)$ does not vanish between x_1 and the root. When $f''(x)$ is zero or infinite at the root, x_1 should be taken sufficiently near the root. Even when $f'(x)$ is infinite at the root, Newton's formula succeeds at times.

The general conic

By M. V. JAMBUNATHAN, *Mysore University,*

The usual method of obtaining the centre and the axes of a conic, say,

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

is to employ a change of axes, comprising of, firstly, a shift in the origin and secondly, a rotation of the axes. According to this method the centre is obtained as the intersection of the two lines

$$ax + hy + g = 0, \tag{2}$$

$$hx + by + f = 0, \tag{3}$$

while the inclination θ of an axis of the conic to the x-axis is given by the relation

$$\tan 2\theta = \frac{2h}{a-b}. \tag{4}$$

Hansraj Gupta has given in *The Mathematics Student*, Vol. XXI, 1953, pp. 56-59, a direct method of deriving the equations to the axes without the transformation of axes, but making use of the intrinsic properties of the conic. The object of the present note is to bring out certain additional results flowing from the method developed by Gupta, as well as to suggest some simplification in the algebra of that method.

The equation to the chord of the conic with middle point at (x_1, y_1) is

$$(x - x_1)(ax_1 + hy_1 + g) + (y - y_1)(hx_1 + by_1 + f) = 0. \tag{5}$$

If the inclination θ of the chord to the x-axis is fixed, so that its slope $t = \tan\theta$ is also fixed, then we must have

$$(ax_1 + hy_1 + g) + t(hx_1 + by_1 + f) = 0. \tag{6}$$

This shows that the equation to the diameter bisecting chords parallel to $y = xt$ is

$$(ax + hy + g) + t(hx + by + f) = 0. \tag{7}$$

From this it is at once seen that (2) and (3) are the equations to the two diameters corresponding to $t = 0$ (or $\theta = 0$) and $t = \infty$ (or $\theta = \pi/2$) respectively; line (2) bisects chords parallel to the x-axis, and line (3) bisects chords parallel to the y-axis. The centre (α, β) is obtained as the intersection of these two diameters.

The method, it will be noted, furnishes an explanation as to how the lines (2) and (3) are related to the conic (1), a feature that is lacking in the traditional method.

Since (7) will be an axis of the conic if and only if it is perpendicular to the chords which it bisects, it follows that if $h = 0$, then line (2) is perpendicular to the x-axis, and line (3) is perpendicular to the y-axis, so that the axes of the conic are $ax + g = 0$, $by + f = 0$; that is to say, the axes of the conic are parallel to the co-ordinate axes. When $h \neq 0$, the line (7) will be at right angles to the line $y = xt$ if t satisfies the condition

$$ht^2 + (a - b)\,t - h = 0. \tag{8}$$

There are thus two values of t in general, corresponding to the two axes. Denoting these two values by t_1 and t_2, we note that

$$t_1 + t_2 = -\frac{a - b}{h},\ t_1 t_2 = -1, \tag{9}$$

indicating that the directions of the two axes are mutually perpendicular. Substitution of these values of t in (7) gives the equations to the two axes separately. Since $t_1 t_2 = -1$, the two equations resulting from (7) may be written as

$$(ax + hy + g) + t_1 (hx + by + f) = 0,$$

and

$$(ax + hy + g) - 1/t_1 (hx + by + f) = 0.$$

If the values of θ corresponding to the values t_1, t_2 of t be θ_1 and θ_2 respectively, then it is shown that

$$2 \cot 2\,\theta_1 = \cot\theta_1 - \tan\theta_1 = \frac{1}{t_1} - t_1 = -t_2 - t_1 = \frac{a - b}{h}.$$

This last expression is also the value of $2 \cot 2\theta_2$. Therefore, θ, the inclination to the x-axis of an axis of the conic satisfies the relation

$$2 \cot 2\,\theta = (a - b)/h,$$

which leads to the well-known relation (4).

It is worth noting that starting from equation (4) and substituting $2t/(1 - t^2)$ for $\tan 2\theta$, we may arrive at equation (8). This provides

a kind of a connecting link between the different results of the two methods.

Having obtained the values of θ_1 and θ_2, the lengths of the axes may be obtained from the equation

$$r^2 (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta) + s_1 = 0,$$

or

$$r^2 \frac{1 + t^2}{a + 2ht + bt^2} + (g\alpha + f\beta + c) = 0.$$

Therefore

$$r^2 = - (g \alpha + f \beta + c) \{(a + 2ht + bt^2)/(1 + t^2)\}.$$

The two values of t, on substitution in this equation, yield the two values of r^2, the squares on the semi-axes.

Equation (8) could also be used to find the equation to the bisectors of the angles between two lines when their combined equation is given in the form (1). Regard these bisectors as diameters that bisect at right angles ' parallel chords ' of the pair of lines ; then the slopes of these bisectors are the roots of t in equation (8), that is, $(a - b)/h = (1 - t^2)/t$. But $t = (y - \beta)/(x - \alpha)$, where (x, y) is any point on either bisector, and the point (α, β) the intersection of the two lines as determined by (2) and (3). Eliminating t, the equation to the pair of bisectors is

$$\frac{(x - \alpha)^2 - (y - \beta)^2}{(x - \alpha)(y - \beta)} = \frac{a - b}{h}. \tag{10}$$

By employing the above method it is also possible to obtain the equations to the asymptotes of the hyperbola. An asymptote of a hyperbola may be regarded as a diameter of the conic and possessing a particular property. Focussing our attention on the angle at which a diameter meets chords which it bisects, we note that the diameter becomes an axis when this angle is a right angle while the diameter becomes an asymptote when the angle assumes the limiting value 0. Regarding this as the characteristic property of an asymptote, it may be asserted that (7) will be an asymptote if and only if it has the same slope as the line $y = xt$. The condition for this is that

$$(a + ht) + t(h + bt) = 0 \quad \text{or} \quad bt^2 + 2ht + a = 0.$$

This equation in t yields two roots t_1 and t_2, which give separately the slopes of the two asymptotes. The equation has *real* roots only when $(ab - h^2)$ is negative, which is the condition that the conic should be a hyperbola. Further, we have the relations

$$t_1 + t_2 = -2h/b, \; t_1 t_2 = a/b,$$

showing that the asymptotes are at right angles if $a + b = 0$. In fact, the angle between the asymptotes is given by

$$\tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} = \tan^{-1} \frac{\pm 2\sqrt{h^2 - ab}}{a + b}.$$

The combined equation to the two asymptotes is

$$b(y - \beta)^2 + 2h(x - \alpha)(y - \beta) + a(x - \alpha)^2 = 0,$$

which reduces to

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0, \tag{11}$$

where $k = -(g\alpha + f\beta)$. The above method is simpler than the usual method employed in text-books for obtaining the equations to the asymptoes of a hyperbola.

An application of projectivities

By Mohindar Singh, *National Defence Academy, (Dehra Dun)*

Statement. *Given any three conics in a plane, in general, there are three points on any given straight line whose polars with respect to those conics will be concurrent.*

In what follows $O(p)$ will denote a pencil with vertex O and ray p. We will first prove another interesting

LEMMA. *If $O_1(p_1)$ is projective with $O_2(p_2)$ and $O_2(p_2)$ is projective with $O_3(p_3)$, then in general, three triplets of corresponding rays of these pencils are concurrent.*

The pencils $O_1(p_1)$ and $O_2(p_2)$ are given to be projective. Therefore their product is a conic C_1 passing through O_1 and O_2. Similarly $O_2(p_2)$ and $O_3(p_3)$ are given to be projective and therefore their product is also a conic C_2 through O_2 and O_3. Let these two conics intersect in L, M, N besides O_2. Now consider L. L is the intersection of corresponding lines p_1, p_2 of the first two pencils. As L also lies on the second conic and therefore p_2 and the corresponding line p_3 in the third pencil also meet in L. Hence corresponding to each point of intersection of these two conics there is a triplet of corresponding lines which are concurrent except possibly the one arising from O_2.

Consider O_2. O_1O_2 in $O_1(p_1)$ corresponds to the tangent at O_2 to the conic C_1. Also $O_3\, O_2$ in $O_3(p_3)$ corresponds to the tangent to C_2 at O_2. Also if O_1O_2 and O_3O_2 are corresponding lines there must be a unique corresponding line in $O_2(p_2)$ and the two tangents quoted above will be coincident. In other words the conics C_1 and C_2 will touch at O_2 and the points L, M, N will reduce only to two points giving us ultimately three points of intersection giving triplets of concurrent corresponding lines.

It should be noted further that the pencils $O_1(p_1)$ and $O_3(p_3)$ are also projective, their product being another conic C_3 passing through O_1 and O_3. We find therefore that this third conic will pass through the points L, M, N and O_1, O_3 and it intersects each of the other conics in four points which do not give rise to any additional triplet of concurrent corresponding lines in the three pencils.

The following special cases arise :

(1) If O_1O_2 is a self corresponding line in the projective pencils $O_1(p_1)$ and $O_2(p_2)$ then the conic C_1 will break up into a pair of straight lines namely O_1O_2 and another line. C_2 will cut it in

four points, two of them lying on O_1O_2 and the other two on the other line of the pair. Now the points on O_1O_2 do not give us distinct triplets of distinct concurrent lines. Hence corresponding to the other points of intersection, we will have two triplets of such concurrent corresponding lines.

(2) If O_1O_2, O_2O_3 are both self corresponding rays, then both C_1 and C_2 will break into pairs of straight lines and the number of points of interesection not lying on O_1O_2 or O_2O_3 will be one only.

Let us apply the above lemma to the proof of the problem stated at the top.

Let the given conics be C_1, C_2 and C_3 and let o be any given straight line. Let O_1, O_2, O_3 be the poles of o with respect to C_1, C_2, C_3 respectively. Obviously, there is a (1, 1) correspondence between any point P on o and its polars p_1, p_2, p_3 w.r.t. the three given conics thus establishing a mutually projective relationship between the pencils $O_1(p_1)$, $O_2(p_2)$, $O_3(p_3)$.

By the application of the lemma there are, in general, three triplets of corresponding rays of these pencils which are concurrent. Therefore there exist three points on o, namely, the three poles of the concurrent triplets of the pencils $O_1(p_1)$, $O_2(p_2)$, $O_3(p_3)$, whose polars w.r.t. C_1, C_2 and C_3 are concurrent.

The following special cases arise :

(i) If the line o passes through a vertex of a self-conjugate triangle of C_1, C_2, or C_2, C_3, or C_3, C_1, one of the lines O_1O_2, O_2O_3, O_3O_1 will become a self-corresponding ray of the projective pencils, reducing the number of points on o whose polars w.r.t. C_1, C_2, C_3 are concurrent to two.

(ii) If the line o is a side of the self-conjugate triangle of C_1, C_2, or C_2, C_3, or C_3, C_1 two of the lines O_1O_2, O_2O_3, O_3O_1 will become self-corresponding rays reducing the required number of points on o to one.

Conics

By A. N. Srinivasa Raghavan

The object of this note is to supply a logical two dimensional definition of a conic, as the focus-directrix definition fails to cover certain limiting cases.

In what follows a, b, e etc. have their usual meaning.

Taking the major axis AA' ($= 2a$) of the ellipse fixed and beginning with the biggest ellipse (circle on AA' as diameter), in the limit when the semi-minor axis $b = 0$, the ellipse degenerates into two coincident finite straight lines AA', eccentricity $e = 1$, because $a^2(1 - e^2) = b^2$ throughout.

Again taking the real axis $A'A$ ($= 2a$) of the hyperbola fixed, when $b = 0$, the hyperbola reduces to a pair of two coincident semi-straight lines AX and $A'X'$. Here again $e = 1$

Taking AX for the axis of the parabola, as the focus S approaches A the curve becomes narrower and narrower and in the limit when S coincides with A the parabola reduces to two coincident semi-infinite straight lines AX.

From the above, it would be seen that the degenerate case of an ellipse or a hyperbola is different from that of a parabola.

Now according to the focus-directrix definition, a conic is the locus of a point the ratio (e) of whose distance from a fixed point to that from a fixed straight line is constant. Also when $e = 1$ the curve is a parabola, when $e < 1$ the conic is an ellipse and when $e > 1$, the conic is a hyperbola. This definition does not cover the degenerate cases considered above.

Again let us take the straight line PQ as directrix and a point S on PQ as the focus. Then when $e = 1$ the locus is the whole of the infinite straight line through S perpendicular to PQ. This is certainly not a parabola which can only be a semi-infinite straight line drawn

from S, perpendicular to PQ in one of the directions and not both. This infinite straight line (or to be more accurate the two coincident infinite straight lines) is really the limiting case of hyperbola when $a^2 = 0$, $b^2 = 0$, and $e = 1$.

Thus it has been established that the focus-directrix definition fails to satisfy the requisite conditions for a logical definition. The definition here suggested is based on the following well-known

THEOREM. *The locus of the centre of a circle touching two given circles is a pair of conics with the centres of the two circles as foci and eccentricities* $d/(r_1 + r_2)$ *and* $d/(r_1 \; r_2)$, *where* d *is the distance between the centres of the given circles and* r_1 *and* r_2 *are their radii.*

Thus a conic may be defined as the locus of a point equidistant from a given circle and a given point (point circle).

This definition is not only simple but is also logically the most natural one. The locus of points (i) at constant distance from a given point, (ii) at a constant distance from a given straight line, (iii) equidistant from two given points and (iv) equidistant from two given straight lines, should logically and naturally be followed by the locus of points equidistant from (i) a point and a straight line and (ii) a point and a circle.

In the foregoing definition when the given point S is inside the given circle, the locus of the moving point P is an ellipse and in this case the distances are equal in magnitude but opposite in sign. [The assumption is that the direction away from the centre is positive and that towards the centre is negative, distances being measured normally from the circumference.]

When the given point S is outside the given circle, the locus of the moving point P is a hyperbola and in this case the distances are equal in magnitude and sign.

The limiting case of the hyperbola (the ellipse) is obtained by moving the given point S from outside (inside) the given circle to its circumference. We get a parabola when the given circle becomes a straight line.

QUESTIONS FOR SOLUTION

QN. 1843. (L. C. HSU). *Let* $a_1, a_2, \ldots, a_n$ *be* n *positive numbers, not all equal, and denote*

$$M_r(a) = \left(\frac{1}{n}\sum_{k=1}^{n} a_k^r\right)^{1/r}, \quad (r \neq 0).$$

Prove that

$$1 < \frac{M_s(a) - M_{-s}(a)}{M_s(a) - M_0(a)} < n,$$

where s *is any positive number and* $M_0 = \lim_{r \to 0+} M_r = (a_1\, a_2 \ldots a_n)^{1/n}$ *is the geometrical mean of* a_k*' s.*

QN. 1844. (L. C. HSU). *Let* $a_1, a_2, \ldots, a_n$ *be* n *positive numbers, not all equal, and let* $r > s > t > 0$. *Prove that*

$$1 < \frac{M_r(a) - M_t(a)}{M_r(a) - M_s(a)} < \frac{s(r-t)}{t(r-s)},$$

where the upper bound on the right-hand side is best possible.

QN. 1845. (VICTOR THEBAULT). *Given a plane quadrilateral there exists an inscribed conic which has the Newton line of the quadrilateral for its focal axis. Let* F, F' *be the real foci of this conic. The 8 points of contact, with the sides of the quadrilateral, of the two conics* (F), (F') *which have* F *and* F' *respectively as their centres lie on a circle.*

NEWS AND NOTICES

The following persons have been elected to life-membership of the Society : S. K. Lakshmana Rao, R. S. Misra.

The following persons have been elected to membership in the Society : V. Cadambe, H. Chaudhary, P. C. Consul, F. W. Hoagland, V. D. Phadke, L. V. Subramanian, R. C. Varma.

The twenty-first Conference of the Indian mathematical Society is to be held in the last week of December 1955, in Banaras, under the auspices of the Banaras Hindu University. Those intending to read papers at the Conference are requested to send the abstracts, not exceeding 300 words, to Professor K. Chandrasekharan, Tata Institute of Fundamental Research, Bombay 1, on or before the 15th of November 1955.

The first Congress on Theoretical and Applied Mechanics will be held on November 1-2, 1955 at the Indian Institute of Technology, Kharagpur.

A Conference on Mathematical Education in South Asian Countries will be organized in February 1956 by the Tata Institute of Fundamental Research, with the support of Unesco, the International Mathematical Union and the National Committee for Mathematics in India. The Conference is expected to discuss in detail problems concerning mathematical education, at all levels, and formulate plans for its sound development, keeping in mind its relationship with education in science in general. The cooperation of every university in the country will be sought for this purpose.

The preparatory work for the conference is proposed to be carried out by a number of panels. The convener of each panel will receive memoranda from other members of the panel as also from any other interested mathematicians, on the specific matters that fall within the purview of the panel, and will prepare a draft report on the work of the panel. With the reports of all the panels in hand, a

programme is proposed to be drawn up. A limited number of mathematicians from India and from other South Asian countries will be invited to present papers, while a larger number of mathematicians will be invited to attend the Conference. Facilities are sought to be provided for their attendance. A report on the Conference is also proposed to be published in due course.

Three panels have been constituted, one for post-graduate teaching and research with Professor K. Chandrasekharan, Professor of Mathematics, Tata Institute of Fundamental Research, Apollo Pier Road, Bombay 1 as Convener, one for graduate education with Professor C. Racine, Professor of Mathematics, Loyola College, Madras 6 as Convener and one for undergraduate education with Professor K. R. Gunjikar, 17 Mathew Road, Bombay 4 as Convener.

Mathematicians who are interested in communicating their views to the Conference are requested to correspond with the conveners of the concerned panels without delay since the final programme will soon have to be drawn up.

On the invitation of the British Council, and the Universities of London, Cambridge, Manchester, Birmingham, Durham and Leeds, Professor K. Chandrasekharan will be visiting England in October 1955. The visit is arranged under the Commonwealth University Interchange scheme for "scholars of distinction actively engaged in academic work."

On the invitation of the Muslim University, Aligarh, Dr. V. Ganapati Iyer delivered at Aligarh a course of five lectures on 'Functional analysis and Linear space of entire functions'.

The executive Committee of the International Commission for Mathematical Instruction has been constituted as follows: H. Behnke (*President*), D. Kurepa and M. H. Stone (*Vice Presidents*), Ram Behari, K. Piene and E. A. Maxwell (*members*).

Dr. R. Vaidyanathaswamy has been appointed Professor and Head of the Department of Mathematics, Sri Venkateswara University, Tirupati.

T. V. Avadhani has been awarded the Ph.D. degree by the Andhra University, Waltair for his thesis '*Eigenfunction expansions.*'

Mansoor Ahmed has been awarded the Ph.D. degree by the Muslim University, Aligarh for his thesis '*Entire functions and Cauchy's theorem.*'

We regret to report the death of the following members of the Society : Prof. G. V. Bhagwat, Bombay ; Rev. E. de Rafael, St. Xavier's College, Bombay ; Dr. V. P. Venkatachari, Professor of Mathematics, Osmania University, Hyderabad (Deccan) ; and Dr. T. Vijayaraghavan, Director, Ramanujan Institute of Mathematics (Karaikudi), Madras.

THE PEDAGOGICAL IMPORTANCE OF HUNTINGTON'S AXIOMS FOR THE COMPLEX NUMBER SYSTEM

By P. MASANI

I. Postulation versus Construction

1. **Introduction.** Our purpose is to point out the desirability of abandoning in calculus, analysis and algebra courses, the constructive treatment of the real and complex numbers such as is given in Chapters I and III of Hardy's *Pure Mathematics* [6], and to suggest Huntington's axioms for the complex numbers [8, 9] as the basis of an alternative treatment.

From the pedagogical standpoint, a constructive treatment has many shortcomings (§2), which are not found in a postulational approach following Huntington's ideas (§3). In Part II, (§4), we shall show in some detail how this approach works out in a course on elementary analysis or advanced calculus. We shall also explain (§5) how the same approach can be adapted to the needs of a more elementary course, such as the First Year algebra course of an Indian college or the equivalent course of a British secondary school. We will thus show how this approach provides a co-ordinated method of presenting numbers to different classes, which entails as little unlearning and philosophical misconception as possible.

2. **Shortcomings of the constructive treatment.** An obvious drawback in defining real numbers in terms of rational numbers is the extreme tediousness of the derivation of the properties of the former, so that one is forced to slur over most of it in a calculus or analysis course. Time and effort have to be spent, however, on the logical niceties that are entailed, e.g. on the difference between an ordinary rational number and the corresponding real rational number, and these are wasted as such niceties are of no importance in the later development of the subject.

But the question of the introduction of the real numbers goes deeper, and is tied up with the controversy over construction and postulation. In 1919 Russell [13, p. 71] spoke of the former as "honest toil", and of the latter as "theft", but we now know this to be a somewhat oversimplified attitude [14, preface to 2nd ed.]. The starting point of logistic construction has to be axiomatic, and the controversy reduces to a disagreement on the choice of primitive terms and axioms [2, §84; 3, §20]. Logisticians would suggest that only a few very fundamental ideas and propositions be taken as primitive, and all would agree that mathematicians should not indulge in *ad hoc* postulation. But what constitutes a "fundamental" idea, and whether it is advisable to take as primitive ideas those far removed from the subject at hand, are questions, which can only be settled by the extra-logical considerations of scope, convenience and taste.

To what extent can the class-room presentation of numbers be based on constructivist principles? Clearly there is no merit from a constructivist standpoint in taking for granted a system as complex as that of the rational numbers, and even so slurring over many important steps in the subsequent construction. On the other hand, it would be even more difficult and tedious in the class-room to adopt the logistically sounder policy of starting with the integers and basing the number-theoretical development on the Peano axioms, as Landau has done [11].

There thus seems to be no constructivist treatment of real numbers, really suited to the kind of course we have in mind, and we are led to the alternative of directly postulating the properties of such numbers or even of the complex numbers. Landau has objected to this alternative on the ground that it obscures the questions of consistency and independence [11, preface]. But as the work of Gödel has shown, the question of consistency is very difficult, and hence its discussion is best reserved for a special course on mathematical logic. Moreover, the consistency of Huntington's algebraic axioms, relative to those of set-theory or geometry, is well known [8, 9]. As such relative-consistency proofs are all we can hope for,

qualms about using his axioms on the score of possible inconsistency are unjustified. Finally, too much preoccupation with the question of independence entails a loss of both interest and direction.

We shall now approach this question of the class-room presentation of numbers from a different angle, which will also clearly point in favour of direct postulation. In the development of a deductive science, the following stages are discernible :

(1) semantical study of a model,

(2) postulational treatment,

(3) metamathematical analysis.

In geometry, for instance, the *model M* is the set of physical objects in whose geometrical features we are interested. In studying M we use technical terms such as "point", "line", etc. but these are endowed with concrete meanings, e.g. a point is a pin-head, a line a stretched thread, etc. The statements regarding M, which are found to be approximately true, constitute a *semantical system S* [3, 4], and their study would come under (1) above. We then decide to make a deductive system out of S by treating certain statements of S as postulates—usually after considerable refinement and idealization—and deducing other truths from them. Logical deduction being a formal process [2, §1], the meanings of the words "point", "line", etc. now cease to be important, and their usage comes to be governed entirely by the postulates containing them. The logical terms occurring in the postulates, e.g. "not", "and", "all", "is", are governed by their usage in scientific English. The semantical system S thus gives way to an *axiom system A* with "point", "line", etc. as *specific primitive terms* [3, §16]. By deduction in A we can settle ordinary geometric questions, e.g. we can prove the similarity of equi-angular triangles. Such study would fall under (2) above. Finally, we may ask questions of a different type, e.g. "Is A categorical?", "Is A equipollent to the Peano axiom system?", etc. These pertain to the axiom-system A as a whole, and since they can be formulated only in a metalanguage, their study would belong to (3) above [4, §1].

To return to numbers, we observe the use made of integers and fractions in daily life, commerce and industry, and of numbers such as $\sqrt{2}$, π in the mensuration of triangles, rectangles and circles. An initial informal study of numbers based on their empirical meaning would belong to stage (1) above. The sentence "there are 5 apples in this bag", for instance, would mean : on counting, we can pair-off each of these apples with just one of the words "one", "two",..., "five". Many arithmetical rules can be found from such a study.

The next logical step would be to treat some of these rules, or rather suitable refinements of them, as postulates and to derive others from them. But the history of arithmetic does not follow this path. On the contrary, different categories of numbers are introduced at different times to cope with the inverse operations of subtraction, division and root extraction, and attempts to define a new category in terms of the old precede postulation. For instance, Dedekind's definition of a real number [5] (1872) preceded the first axiom system for the real numbers (Hilbert), [7], (1900).

We must emphasise, however, that the question :

(4) *Are real numbers definable in terms of rational numbers?* is meaningful only if the usage of the terms "real number", "rational number" is specified beforehand. Actually, 19th century mathematicians did use these words in a definite sense. Dedekind, for instance, would have characterized the rational numbers as an ordered field devoid of proper subfields, and the real numbers as an ordered field having "continuity" [5, pp. 4-5, 10-11]. He would have paraphrased the question (4) essentially as follows :

> The development of the calculus by Cauchy, Weierstrass, Heine and others can be made rigorous if the number system they employ can be shown to have the properties of *field* and *continuity*. Are sets of rational numbers and relations between such sets definable, which will form a system exhibiting these properties ?

Before tackling question (4), we must therefore mathematically characterize the rational and real number concepts. Now the only way of doing this is to write down a set of sentences containing

symbols for such numbers. These sentences, being the usage-specifying conventions for these symbols, would in reality be axioms for them. Question (4) thus presupposes an axiomatization of the rational and real number concepts, and belongs to the metamathematical stage (3) above, its precise rendering being :

Let A be an axiom system for the real numbers and B one for the rational numbers. Can the specific primitive terms of A be defined in B† so as to convert the axioms of A into theorems or axioms of B?*

Our task in the class-room before posing question (4) is thus to give an axiom system for the real numbers. The next step would be to do calculus proper and show thereby that the real number concept, so axiomatized, suffices for the purposes of the calculus—as indeed Cauchy, Heine and Weierstrass had shown unwittingly before Dedekind. Question (4), which belongs to the province of logistics rather than calculus proper, can thus be circumvented or rather reversed ; for the problem now is to explain the integers and rational numbers as special types of real numbers.

Similar remarks apply to the transition from the real to the complex numbers. The question "are complex numbers definable in terms of real numbers?" presupposes an axiomatization of the complex number concept, and once this axiomatization is made in the class-room, the question can be avoided altogether. The problem is rather to characterize the real numbers as special sorts of complex numbers.

3. A better treatment. The moral of the preceding discussion is clear : notwithstanding the importance of the logistic method, we should wait a little longer than we now do before thrusting it on the student. Instead, we should constantly keep before the student the

*e. g. *A* may be the system given by Hilbert [7]. *B* may be any set of axioms for an ordered field not having proper subfields.

†i. e. can each specific primitive term of *A*, or an expression containing it, be treated as an abbreviation of an expression of *B* ?

model the number system is intended to serve, and lay down axioms as and when the need for each is felt. For this purpose the axioms due to Huntington, paraphrased in terms of *fields* and *ordered fields*, and in which C refers to the set of complex numbers, are ideal:

AXIOM 1. *C is a field.*

AXIOM 2. *C contains an ordered subfield R.*

AXIOM 3. *Every subset of R, which is bounded above, has a least upper bound.*

AXIOM 4. *C has a member i such that $i^2 = -1$.*

AXIOM 5. *Every member of C is of the form $x + iy$, where $x, y \in R$.*

Though Hardy's *Pure Mathematics* is a work of outstanding merit, its treatment of the number concept needs revision. We feel that a course covering this book could very well be based on the suggested treatment of numbers. Another feature of this book, also in need of modernization, viz. the introduction of the function-concept, has been dealt with elsewhere [12].

II. Class Room Presentation of Numbers

4. **Presentation in calculus courses.** We shall now describe the actual presentation of numbers from the standpoint explained in §2, Part I, to a class in elementary analysis or advanced calculus. Our presentation before such a class breaks itself up into the discussion of (A) ordered fields, (B) the number-concept, (C) integers, (D) rational numbers, (E) real numbers, and (F) complex numbers. It entails an initial discussion of abstract concepts and intermittent discussion of the choice of axioms, but is more economical on the whole than other treatments.

A. *Ordered fields.* We begin by briefly explaining the concept of a set or class, and defining the expressions

$$a \in A, \quad A = B, \quad A \subseteq B.$$

We give examples to show that relations often prevail among the members of a set, and proceed to study sets which exemplify the four basic operations of arithmetic: addition, subtraction, multiplication, and division. We thus define a *field.*

We then prove the following basic results concerning fields.

4. 1. THEOREM. *In any field*

(a) $a0 = 0$;

(b) $ab = 0$ *implies* $a = 0$ or $b = 0$;

(c) $a + b = a + c$ *implies* $b = c$;

(d) $ab = ac$ & $a \neq 0$ *implies* $b = c$;

(e) $(-a)(-b) = ab$.

For proofs we would refer the reader to any book on modern algebra, e.g. [1]. Most students are glad to know these proofs, especially that of (e). One of the conditions in the definition of a field is that *every non-zero element a has a reciprocal* $1/a$ *such that* $a(1/a) = 1$. If 0 had a reciprocal $1/0$, it would follow from this condition and 4.1 (a) that

$$1 = 0. (1/0) = 0,$$

in contradiction to another defining condition of the field, viz. $1 \neq 0$. We thus establish the following triviality, primarily to destroy wrong ideas of infinity.

4. 2. COROLLARY. *In a field* 0 *has no reciprocal.*

We next refer to the *ordering relation* that subsists between ordinary numbers, and explain how this suggests the following definition.

4. 3. DEFINITION. *We call A an 'ordered set', if a relation* $<$ *is defined on it, satisfying the following conditions : for all* $a, b, c \in A$,

(i) $a \not< a$,

(ii) *$a \neq b$ implies either $a < b$ or $b < a$,*

(iii) *$a < b$ and $b < c$ implies $a < c$.*

We also point out that for ordinary numbers the algebraic operations are linked up with the ordering relation, e.g. if $a < b$, then $a - b < 0$, and suggest to the class a combined study based on the following concept.

4. 4. DEFINITION. *An 'ordered field' is a set F, which is a field as well as an ordered set and in which the binary operations $+, \cdot$, and the ordering relation $<$ are connected as follows: for all $a, b, c \in F$,*

(i) *$a < b$ implies $a + c < b + c$*

(ii) *$0 < a, b$ implies $0 < ab$.*

We then establish the following simple results on ordered fields (cf., e.g. [1, Ch. II]).

4.5. THEOREM. *In any ordered field,*

(a) *the conditions $a < b$, $0 < b - a$, $-b < -a$ are equivalent,*

(b) *$0 < a < b$ if and only if $0 < 1/b < 1/a$,*

(c) $0 \leqslant a_1^2 + \ldots + a_n^2$,

(d) *$0 = a_1^2 + \ldots + a_n^2$ implies $a_1 = \ldots = a_n = 0$,*

(e) $0 < 1$, $0 < 1 + 1$, *etc.*

Finally, we define *subfields*. This is all the abstract algebra that we do in class.

B. *What is a number?* We next raise this question, and point out that it may be approached in one of two ways (i) the *constructive* or *logistic way*, in which we define a number in terms of *set, relation* and other fundamental logical notions, (ii) the *axiomatic* or *postulational way*, in which we lay down a basic set of sentences involving notions such as $+, \cdot, <$, called *axioms*, and indirectly define a number as the member of a set satisfying these axioms. We tell the class that (ii) is the simpler approach, and will be followed. For the benefit of the keen students, we add that the approach (i) is also

ultimately axiomatic, and that the difference between (i) and (ii) lies in the choice of primitive concepts and axioms.

We then ask the class as to what axioms we should take. We emphasise the fact that the axioms must yield numerical results needed in everyday life, engineering, science, etc. but that apart from this they may be chosen in any convenient and consistent way. We recall the practical origins of Euclidean geometry to illustrate this point. The geodetical study of the Nile basin provided an approximate model for Euclid's geometry and in fact suggested the axiomatization; on the other hand, the geometrical axioms given by Hilbert and others can be used to deal with the same model.

As we wish to be able to perform the four arithmetical operations with numbers, we take the following as our first axiom, C being the set of numbers :

4.6. Axiom 1. *C is a field.*

It is to be understood that the specific primitive terms of our axiom system are '$+$', '$\cdot$', '0', etc., and that 4.6 is only an abbreviated way of writing the field postulates. Similar remarks apply to the axioms which will follow. We do not, however, refer to this point in the class-room.

We remind the class of the ordering relation, and ask as to how this can be introduced in C, in the hope that some one would suggest that we should assume C to be an ordered field. We promptly demolish this suggestion by referring to 4.5, and pointing out that such an axiom would debar us from solving the quadratic equation $x^2 + 1 = 0$, and thus become a liability. As a compromise we adopt the following axiom.

4.7. Axiom 2. *C contains an ordered subfield R.*

We then stop giving axioms, and turn to the classification of the members of C on the basis of the two already given.

C. *Integers.* We define a *whole number* as any member of C of the type 1, $1 + 1$, etc. but also give the following recursive definition, primarily for the benefit of the rigorously minded.

4.8. DEFINITION. *The set of whole numbers is the smallest subset N of C such that*

(i) $1 \in N$, *and* (ii) $x \in N$ *implies* $x + 1 \in N$.

The most fundamental theorem on the whole numbers, viz. the *principle of induction*, is just a paraphrase of our definition of whole number. This is one advantage of the approach we are outlining. From this principle we easily deduce the following result.

4.9. THEOREM (a) *The sum and product of whole numbers m, n is a whole number.*

(b) *For every whole number n, $1 \leqslant n$.*

We turn next to the integers, defining them as follows.

4.10. DEFINITION. *We call $x \in C$ an integer, if an $x = 0$ or $x = \pm n$, where n is a whole number.*

From 4.9 (a) and 4.10 we readily infer :

4.11. COROLLARY. *The sum, difference, and product of integers m, n is an integer.*

It follows from 4.9 (b) and 4.5 (a) and (e) that if n is a whole number, then $-n \leqslant -1 < 0$. A positive integer cannot therefore be the negative of a whole number. We thus prove the following corollary.

4.12. COROLLARY. *x is a positive integer if and only if it is a whole number.*

D. *Rational numbers.* We define these as *members of C of the form m/n, i.e. $m(1/n)$, where m, n are integers and $n \neq 0$.* Since $n = n/1$, every integer is a rational number. We show next that if we put '2' for '$1 + 1$', then by 4.5 (e) and 4.4 (i), $0 < 1 < 2$; hence by 4.5 (b), $0 < 1/2 < 1$, and therefore by 4.9 (b) and 4.12, $1/2$ is not a positive integer. Since $1/2$ is positive, it cannot be any kind of integer. Rational numbers, which are not integers, we call *fractions*. We then sketch the proof of the following important result.

4.13. THEOREM. *The rational numbers form an ordered subfield R_0 of R.*

We then turn to the deficiencies of R_0. We emphasise the result that there is no rational number whose square is 2, and reveal its extension, viz. *if the integer a is not a perfect n-th power, then there is no rational number x such that $x^n = a$.* If the fundamental theorem of arithmetic is not done earlier, we must be content with an informal proof of this result.

We also interpret this deficiency in terms of the one-one correspondence between the rational numbers and the points of a Euclidean straight line, by constructing points not corresponding to any rational number. In so far as Euclidean geometry is treated informally, this part of the work is necessarily devoid of rigour. Stated otherwise, we show that with a given segment as unit it is impossible to measure the lengths of all other segments by means of rational numbers alone.

E. *Real numbers.* We thus confront the class with two alternatives: (i) to give up the idea of extracting n-th roots in most cases, and of measuring the lengths of all segments with a given unit segment, or (ii) to impose a new axiom on C, which will ensure that R has members other than rational numbers, i.e. that $R \supset R_0$. We naturally choose the second alternative.

We point out that to arrive at a suitable axiom, it is necessary first to exhibit some general feature of R_0 to which its deficiencies can be ascribed. We then introduce the notions of a *bounded subset,* the *least upper bound* (l. u. b.) and the *greatest lower bound* (g. l. b.) for any ordered set (cf. 4.3). We show that the absence of a rational number x such that $x^2 = 2$ is related to the fact that *the bounded set S of positive rational numbers x, such that $x^2 < 2$, has no l. u. b.* We actually prove (cf. Hardy [6]) that S has no maximum and the set of its upper bounds has no minimum. We state, rather dogmatically, that each deficiency of R_0 may be attributed in this way to the existence of a bounded subset, not having a l. u. b.

Recalling that $R_0 \subseteq R$, we do all we can to goad the class into suggesting the following axiom.

4.14. AXIOM 3. *Every subset of R, which is bounded above, has a l. u. b.*

We now define a *real number* as any member of R, and using Axiom 3 prove the following theorem.

4.15. THEOREM. *Given a positive real number a and a positive integer n, there exists a unique positive real number x such that $x^n = a$. (We denote x by $\sqrt[n]{a}$.)*

Since $R_0 \subseteq R$, every rational number is of course a real number. But the real number $\sqrt{2}$, whose existence is ensured by 4.15, is not a rational number, for as noted above, there is no rational number whose square is 2. Real numbers, which are not rational numbers, we call *irrational numbers.*

We also mention the fact that the one-one correspondence between the rational numbers and certain points on the Euclidean straight line can be extended into a one-one correspondence between R and the entire line.

F. *Complex numbers.* We now point out that we are still unable to solve the quadratic equation $x^2 + 1 = 0$, by recalling that no ordered field will contain a root of this equation (cf. 4.5 (c)). We again confront the class with the alternatives of giving up the idea of solving all quadratic equations, or admitting in the system C members not belonging to any ordered field, by means of an additional axiom. We naturally adopt the second alternative, and give the following axiom.

4.16. AXIOM 4. *C has a member i such that $i^2 = -1$.*

We next recall the elementary algebraic result that the roots of a quadratic equation with ordinary (i.e. real) coefficients are of the form $a + ib$, where a and b are real. We then state in a rather authoritarian way, that numbers of this form suffice for most mathematical purposes, and that it is therefore desirable to have the following axiom.

4.17. Axiom 5. *Every member of C is of the form $x + iy$, where $x, y \in R$.*

For the benefit of a few keen students, we allude to the fact that Axiom 5 is independent of the earlier axioms, and cite the field of rational functions over R as an example to show this.

Since by Axiom 2, $R \subseteq C$, every real number is a complex number. But the complex number i is not a real number, since $i^2 < 0$. Complex numbers, which are not real numbers, we call *imaginary numbers*. We then prove the following basic result.

4.18. Theorem. *Every complex number can be expressed in the form $x + iy$, where $x, y \in R$, in one and only one way.*

We call x and y the *real* and *imaginary parts* of the complex number $z = x + iy$, and show that z *is imaginary, if and only if* $y \neq 0$.

Theorem. 4.18 shows that there is a one-one correspondence between complex numbers and ordered pairs of real numbers, and therefore between complex numbers and coplanar vectors issuing from a fixed point. We stress this *geometrical representation* of complex numbers in the class-room, and use it to suggest the notions of the *absolute value* and the *amplitudes* of a complex number.

From here on, our treatment of the complex numbers follows the usual pattern. We prove the Cauchy-Schwarz inequality for real numbers and deduce from it the triangle-inequality. As a discussion of the analytical theory of the circular functions [6, Ch. IX] would entail considerable digression, our policy is merely to list the properties of these functions needed to define the amplitude, and to tell the class that these will be established later.

To sum up, our treatment of the real numbers, apart from being concise and to the point, provides a sound basis for a later discussion of the metamathematical questions which naturally arise, and which might appear to a keen student : the isomorphism of all ordered fields satisfying Axiom 3, the consistency of our axiom system, etc. Our class-room policy is to make enough remarks to create awareness of such questions, but not to discuss them.

5. **Presentation in elementary algebra courses.** In such courses we try to proceed as far as possible on the model of the development given in § 4. We do not refer to fields, but taking the idea of number for granted, write down the field postulates treating them as generalizations of arithmetical laws and calling them *basic laws of numbers.* While explaining these laws we make a few historical remarks, e.g. we allude to the qualms once felt by mathematicians about using negatives when explaining the postulate: given any number a, there is a number $-a$ such that $a + (-a) = 0$.

We then tell the class that, as with high-school geometry so here, all algebraic laws can be derived from a set of basic laws. By way of illustration we establish the important laws: $a0 = 0$ and $(-a)(-b) = ab$. But we do not attempt to keep up with a deductive treatment; indeed we do not even write down the postulates for the ordering relation.

We then turn to the classification of numbers. We demarcate the set of *whole numbers* as comprising all numbers of the form $1, 1 + 1, 1 + 1 + 1$, etc. With the aid of many illustrations we try to convince the student of the validity of the principle of induction. We then define the *integers* as comprising the whole numbers, their negatives and zero.

A *rational number* is then defined as one of the form $a\,(1/b)$, where a, b are whole numbers and $b \neq 0$. (The existence of reciprocals is postulated at the outset.) The closure of this category under the four fundamental operations is indicated. The *ordering relation* is introduced by calling a/b positive if ab is a whole number. Important properties of the ordering relation are referred to without proof. Stress is laid on the *geometrical representation.* The non-existence of a rational number whose square is 2 is shown, and this in conjunction with the Pythagorean theorem is used to show that the diagonal of a square is incommensurable with its edge.

We dwell upon this incompleteness of the correspondence between the rational numbers and the points of a straight line, and

try to make the student feel the need for an additional assumption. This naturally has to be stated in a geometrical form.

Assumption. Let p be a straight line and O a point on it. The one-one correspondence (with origin O) between the rational numbers and certain points of p can be extended so that to every point of p corresponds a number.

The number corresponding to a point P on p we call a *real number*, specifically *the directed-distance from O to P* and denote it by "OP". That there are *real irrational numbers* now follows from the incommensurability noted above.

The ordering relation for the reals is defined by means of this geometrical representation : OP is *positive* if and only if P is to the "right" of O on p. Some properties of the real numbers are then discussed without proof, especially the existence of a unique positive n-th root of a positive real number, and the fact that $x^2 \geqslant 0$ for all real x.

The last result is used to suggest to the class the need for further extension for purposes of solving quadratic equations. This is based on the following additional assumption.

Assumption. (a) There is a number i such that $i^2 = -1$. (b) Every number is of the form $x + iy$, where x, y are real numbers.

We conclude with a discussion of the laws of complex algebra. The study of quadratic equations and of exponents and logarithms forms a natural sequel to this treatment of numbers.

6. **Conclusion.** The stages (1) to (3) mentioned in §2, Part I, seem to be inherent in the development of any mathematical science, and in the process of its assimilation. We have clear historical evidence that the earliest geometry was tied up with primitive architecture and mensuration, and that the earliest use of numbers was related to production and exchange [15, Ch. I]. With the progressive evolution of these sciences, attention shifted from the concrete to the abstract, and from isolated rules and techni-

calities to general principles and theory-construction, culminating in postulational treatment. The same trend seems inevitable in the process of learning these subjects. As Klein has put it [10, pp. 3-4]:

> "The teacher so to speak, must be a diplomat. He must take account of the psychic processes in the boy in order to grip his interest; and he will succeed only if he presents things in a form intuitively comprehensible. A more abstract presentation will be possible only in the upper classes. For example : The child cannot possibly understand if numbers are explained axiomatically as abstract things devoid of content, with which one can operate according to formal rules. On the contrary, he associates numbers with concrete images. They are numbers of nuts, apples, and other good things, and in the beginning they can be and should be put before him only in such tangible form."

The treatment of numbers outlined in this paper takes account of this conceptual development (cf. §2). It can accordingly be incorporated in an overall scheme of instruction, extending from primary school to university, in which the study of numbers will form one continuous process, unmarred by unexpected shifts in emphasis and yet revealing the creative side of the subject. At the lowest levels the main task would be to make the pupil discern arithmetical laws in situations involving concrete things, and to familiarize him with their operation. At the next stage the problem would be to formulate these laws in universal form with the aid of variables and quantifiers, and to instil facility in symbolic manipulation. At a still higher stage the emphasis would be on deductive development. And in the final stages, the entire deductive system would be subjected to metamathematical analysis by comparison with other systems.

REFERENCES

1. G. Birkhoff and S. MacLane: *A survey of modern algebra*, New York, 1941.
2. R. Carnap: *The logical syntax of language*, London, 1937.
3. R. Carnap: *Foundations of logic and mathematics*, Chicago, 1939.
4. R. Carnap: *Introduction to semantics*, Harvard, 1942.
5. R. Dedekind: *Essays on numbers*, Chicago, 1909.
6. G. H. Hardy: *A course of pure mathematics*, 10th Ed., Cambridge, 1952.
7. D. Hilbert: Ueber den Zahlbegriff, *Jber. Deutschen Math. Verein.* 8 (1900), 180-184.
8. E. V. Huntington: A set of postulates for ordinary complex algebra, *Trans. American Math. Soc.* 6 (1905), 209-229.
9. E. V. Huntington: The fundamental propositions of algebra, *Monographs on Topics of Modern Mathematics*, edited by J. W. A. Young, London, 1932.
10. F. Klein: *Elementary mathematics from an advanced standpoint*, (*Arithmetic, algebra, analysis*), New York, 1945.
11. E. Landau: *Foundations of analysis*, New York, 1951.
12. P. Masani: What is a function?, *Math. Student*, 19 (1951) 81-101.
13. B. Russell: *Introduction to mathematical philosophy*, 2nd Ed., 1920.
14. B. Russell: *The principles of mathematics*, 2nd Ed., London, 1937.
15. D. J. Struik: *Concise history of mathematics*, Vol. 1, New York, 1948.

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BOOK REVIEWS

Fundamentals of college mathematics. By John C. Brixey and Richard V. Andree, The University of Oklahoma, Henri Holt and Company, New York, XIV, 609 pages.

This text-book deals with a little more than one fourth of the first year of college study at the University of Oklahoma. But, as the authors indicate in their " Preface to the Instructor ", it may be used for shorter courses in trigonometry or analytical geometry or for " terminal courses providing mathematical background desirable for a liberal (or general) education." The same preface informs us that the choice of text and problem material was influenced by interviews and correspondence with many engineers, physicists, chemists, and other scientists. This makes it clear that if the authors' purpose does not exclude the instruction of students intending to proceed to more advanced courses in pure mathematics, it is, in the first place, to train those, no doubt a majority, who will afterwards take to courses of applied mathematics or physics, or chemistry.

It is difficult for one who is not acquainted with the educational system of the United States, to realize how little young people know when they come to college. This text book assumes quite plainly that they know absolutely no mathematics. The first chapter is about the elementary rules of algebra, including explanations about the symbols $\neq$, $<$. The definition of co-ordinates, excluding polar co-ordinates, which one will find only towards the end of the book (in Chapter 17), the elements of calculus, but restricted to polynomials, (the other functions are reviewed only in Chapter 19), the elements of trigonometry and of statistics follow. The last chapters are about the analytical geometry of conics and a very elementary treatment of solid geometry.

More than half of the book comprises examples and exercises, whose answers cover about 100 pages at the end. There is a wealth of sketches, graphs and various illustrations. The student will also find a few allusions to higher mathematics which may spur him

on to read reference books and perhaps create in him a lasting interest in mathematics. In particular the last Chapter, Chapter 25, is entirely devoted to selected topics from more advanced mathematics but merely we are told in the "Preface to the Student" for his (the student's) pleasure.

It is not easy to judge the value of such a text-book. There is no doubt that it will develop in the student a great skill as far as the manipulative side of mathematics is concerned. If it had been written for younger people, I mean for high school boys, there might not be any serious objection to emphasizing the manipulative side of mathematics as the authors have done. But the reviewer finds it difficult to reconcile himself with so superficial a treatment of the main notions, with so little place granted to abstraction. Indeed the authors give the impression that an abstract concept has to be treated with the same awe as a dangerous weapon which nobody can take the responsibility to entrust into the hands of adolescents. If the notion of limit is correctly introduced, that of functions lacks precision and perhaps is it extreme shyness that prevents the authors to apply it to nothing but polynomials until Chapter 19. Students intending to specialize either in physics, engineering, or applied mathematics may probably find this college mathematics a help. Lecturers hunting for examples and exercises will greet it as a blessing. But one wonders whether a young man with a very good disposition for mathematics will not find it a nuisance. A budding mathematician may like tricky manipulations; yet he will, even at a very early age, distinguish the manipulative from the notional side and his mind will be, from the start, more exacting than many mathematics teachers might surmise, about clearness and rigour. He will certainly realize that what matters most is the logical aspect, not the practical. For such promising young men the Fundamentals of College Mathematics may not be a book to be recommended.

C. Racine

Integers and theory of numbers. By Abraham A. Fraenkel, Scripta Mathematica Studies, Number 5, New York, 1955, pp. 102.

This pleasant little monograph is, in the main, a translation of the first part of the author's Hebrew book *Mavo Le Mathematica.* Two more volumes of similar scope and size dealing, respectively, with the fundamental concepts of algebra and the theory of sets and transfinite numbers are promised, and will be eagerly welcomed. The preface states that " these volumes developed from talks in the adult education programme given by the author in towns and rural settlements of Palestine (now Israel) from 1929 on. Consequently, the subject of the present volume and its treatment meet the needs, abilities and interests of gifted high-school students, of college freshmen, and, indeed of laymen who are interested in knowing what mathematics really deals with. " The book could be enthusiastically recommended to junior university students in this country too, and should prove a welcome addition to libraries. It may be usefully translated into Indian languages.

There are four chapters. Chapter I opens with the positional notation and introduces the natural numbers as cardinals, characterizing and characterized by equivalent sets. Chapter II is devoted to natural numbers as ordinals. Peano's axioms are stated and their mutual independence estiblished by constructing contradiction-free ' models ' satisfying all but one of the axioms at a time. The dual function of the axiom of induction as a means of definition and as a method of proof is stressed. The relation of cardinals to ordinals is but briefly mentioned, to be amplified in a subsequent volume. Chapter III, which is the longest in the book, deals with " higher arithmetic ". The topics covered are the infinitude of primes, the prime factorization of integers, sequences of primes (mention is made of Dirichlet's theorem on primes in arithmetic progression, the Goldbach conjecture, etc.), congruences, Fermat's theorem, and perfect numbers. The chapter closes with a very readable account of algebraic and ideal numbers. Chapter IV takes up the construction and properties of the field of rational numbers.

The usefulness of the book is enhanced by an appendix containing, among other things, a proof that a finite set is not equivalent to a proper subset of itself, Zermelo's proof of the uniqueness of the prime power decomposition, and simple proofs of estimates of gaps between consecutive primes, e.g. $p_{n+1}^2 < p_1 p_2 \cdots p_n$ for $n > 3$, $p_{n+1}^3 \leqslant p_1 p_2 \ldots p_n$ for $n > 26$ $(p_1 = 2, p_2 = 3, p_3 = 5, \ldots)$. The inquisitive reader will appreciate the profusion of footnotes containing both supplementary information and indications of further literature.

The book is very well produced and typographical errors are few.

K. BALAGANGADHARAN

An introduction to the theory of numbers. By G. H. Hardy and E. M. Wright, Oxford, 1954, pp. xvi + 419, 42s.

Third edition of the well-known book. For review of the first edition, see *The Mathematics Student*, VI, 1938, pp. 171 to 173. The most significant addition in the present edition is Selberg's elementary proof of the prime number theorem.

K. BALAGANGADHARAN

Popular guide to the heavens. By Sir Robert Stawell Ball, George Phillip & Son Ltd., London, 1955; pp. 84 + xii; 30s.

The book under review deals with one of the most fascinating topics, namely the study of the heavens, which attracted the notice of all peoples since the very dawn of civilization.

The book was first published in 1892 under the title " An Atlas of Astronomy " and it received its present name when its extensively revised edition was brought out by its distinguished author Sir R. S. Ball in 1910. The fourth edition (1925) was brought out

by Dr. T. E. R. Phillips, who preserved the original plan of the book but had to re-write some portions in the light of the data collected during the fifteen years which elapsed since its revision in 1910. The task of bringing out the fifth edition (1952) fell on Dr. E. A. Beet who, with a view to keep the book as "Ball's book" retained its original plan. He has mainly concerned himself with the task of making the book up-to-date in the light of the recent observational data. The observational data in astronomy are growing very rapidly and hence it is really a difficult task to write a book which may be regarded as up-to-date even over a few years. This statement is amply borne out by the addenda given on page viii which bridge the gap between the revision and the publication.

The present edition consists of explanatory notes covering eighty-three pages and an equal number of plates. The explanatory notes are divided into nine chapters : (i) The celestial sphere and the solar system, (ii) The planets and satellites, (iii) The sun, (iv) Comets, (v) The moon, (vi) The sky month by month, (vii) The star maps, (viii) Star clusters and nebulae, and (ix) Select list of telescopic objects. These are written in a lucid, clear and unambiguous language. Consequently even a beginner will not find any difficulty in working through these notes. The most important alterations in this part of the book are the extension of the planet tables so as to give their position upto the year 1970 and the revision of the measures of the double stars.

As far as the plates are concerned, Dr. Beet has included the following in addition to those present in the fourth edition :

(i) Two photographs on the sun-spots of 1946.

(ii) The comet of 1942 (in place of the comet of 1902)

(iii) More pictures of Morehouse's comet.

(iv) A new map of Mars drawn by Mr. Ryves.

The series of plates begins with the plates explaining the equatorial and ecliptic systems of co-ordinates and the phenomena of seasons and tides. These are followed by seven plates dealing with

planets, and nine plates depicting the various solar phenomena, such as spots, flares, prominences, corona, and eclipses. In the last plate of this group, the tracks of the total solar eclipses visible from 1945 to 1970 have been indicated. These are followed by four plates on comets and a number of plates showing the surface features of the moon. We then come to the monthly star maps drawn for the latitude 53° 23′N which are suitable for all parts of the British Isles. These are followed by the general star maps. The entire sky is mapped out in twenty sections on the basis of conical projection. These maps show all stars down to the sixth magnitude and specify the double stars and the variable stars among them. We then come to eight plates dealing with the nebulae which are visible even through small telescopes. The series ends with the map of the world showing the system of standard time adopted on land and sea.

The book in its present form is based on the expert knowledge of three distinguished astronomers. It covers in a systematic manner the heavenly bodies which are accessible to the naked eye or to a small telescope; the plates are authentic and interesting. In the opinion of the reviewer this enlarged and revised edition of Ball's book is bound to be a valuable addition to libraries, and will be welcomed by lovers of astronomy.

P. L. BHATNAGAR

EDITORIAL NOTE

The delay in the appearance of this issue is greatly regretted.

A report of the twenty-first Conference of the Indian Mathematical Society, held at Banaras on 24-26, December, 1955, is published in this issue.

The next issue of the *Mathematics Student* will be a double number for January-April, 1956, of about 150 pages, containing a report on the South Asian Conference on Mathematical Education held at the Tata Institute of Fundamental Research, Bombay, in February 1956. It will appear towards the end of April, 1956.

REPORT OF THE TWENTY-FIRST CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY

THE INDIAN MATHEMATICAL SOCIETY
TWENTY-FIRST CONFERENCE, BANARAS 1956.

LEFT TO RIGHT

Sitting : N. S. Nagendranath, Balak Ram, Hansraj Gupta, G. L. Chandratreya, Eichler, S. Mahadevan, C. P. Ramaswami Iyer, Sampurnanand (*Chief Minister*), Ram Behari, R. Vaidyanathaswamy, R. C. Bhijavat, G. S. Mahajani, B. R. Seth, B. N. Prasad, Brij Mohan.

Standing 1st row : J. Datta, B. V. Radhakumari, S. Sharma, K. Kapila, Suseela, Padmavati, Shyamala, K. Padmavalli, Urmila Devi, Kamala Sen, Gita Haldar, K. Savitri, K. N. Kamalamma, M. Srirangamma, P. Srivastava, S. V. Adavi, M. Parameswara Iyer, R. Shukla, S. Shastri.

Standing 2nd row : Sahib Ram, J. L. Sharma, S. M. Shah, M. V. Subba Rao, T. K. Patil, Ramanand, V. V. Narlikar, K. M. Shah, V. R. Luthara, S. K. Bose, R. S. Mishra, L. N. Chawla, S. N. Kawalgikar, N. Yagyanarayanan, Jugal Kishore.

Standing 3rd row : S. L. Narasimham, S. Moran, A. C. Chaudhary, J. A. Siddiqi, C. R. Rao, M. Raghavacharyalu, P. V. Patel, Labh Singh, V. S. Huzurbazar, Ram Ballabh, S. K. Sharma, O. P. Gupta, B. S. Griwal, P. L. Bhatnagar G. Bandyopadhyaya, N. S. Subbanarayanan, A. S. Sharma.

Standing 4th row : S. C. Das Gupta, B. S. Jain, V. P. Dhar, C. R. Marathe, Harishanker, Ibdur Rahman, B. Das, P. C. Consul, Dinesh Chandra, V. V. L. N. Rao, S. K. Kaul, P. D. Verma, L. V. Subramanian, G. C. Patni, R. C. Prasad.

PROGRAMME

Saturday, 24 December, 1955

10-00 A. M.—Address of Welcome by the Chairman of the Reception Committee, Dr. C. P. Ramaswami Aiyar, Vice-Chancellor, Banaras Hindu University

Inauguration of the Conference, opening of the Mathematical Exhibition and unveiling of the portrait of Mahamana Malaviyajī by Dr. Sampurnanand, Chief Minister, U. P.

Annual Report by the Secretary of the Society, Professor S. Mahadevan

Award of the Narasinga Rao Medal by the President

Address by Dr. R. Vaidyanathaswamy, Professor of Mathematics, Sri Venkateswara University

Vote of thanks by the Local Secretary

11-15 A. M.—Group Photograph : Delegates and Members of the Society

11-30 A. M.—Visit to the Exhibition

1-00 P. M.—Lunch at Delegates' Camp

2-15 P. M.—Business Meeting of the Society

2-30 P. M.—General Meeting

3-00 P. M.—Invited address by Professor M. Eichler, Muenster University, W. Germany

4-00 P. M.—Vice-Chancellor's At Home, Central Hindu College quadrangle

5-00 P. M.—Popular lecture by Dr. G. S. Mahajani, Vice-Chancellor, University of Delhi, on "Extension of the universe in space and time"

8-00 P. M.—Dinner by Maharaj Kumar Dr. Vijay of Vizianagaram, Arts College Hall

Sunday, 25 December, 1955

9-00 A. M.—Reading of papers

11-30 A. M.—Symposium on " Unified Field Theory "

1-00 P. M.—Lunch at Delegates' Camp

2-00 P. M.—Invited address by Professor F. C. Auluck on " Stability problems in Magneto-Hydrodynamics "

3-00 P. M.—Invited address by Dr. K. G. Ramanathan on " Arithmetic over Algebras "

3-30 P. M.—Popular lecture by Dr. P. L. Bhatnagar on 'Our Sun'

4-00 P. M.—Messrs Kohinoor Stores : At Home

5-00 P. M.—Popular lecture by Professor K. S. Krishnan on 'The geometry of our universe'

6-00 P. M.—Variety Entertainment

8-00 P. M.—Dinner by Messrs Pushtak Kutir, C.H.C. Hall

Monday, 26 December, 1955

9-00 A. M.—Reading of papers

11-30 A. M.—Symposium on " Entire and meromorphic functions "

1-00 P. M.—Lunch at Delegates' Camp

2-30 P. M.—Excursion to Sarnath and Tea by Reception Committee at Sarnath

8-00 P. M.—Dinner to Delegates

Tuesday, 27 December, 1955

Excursion to Bathing Ghats

REPORT OF THE CONFERENCE

Inauguration

The Twenty-first Conference of the Indian Mathematical Society met at Banaras on December 24-26, 1955, on the invitation of the Banaras Hindu University. Over a hundred delegates attended.

The Conference opened on December 24 with an address of welcome by Dr. C. P. Ramaswamy Iyer, Vice-Chancellor of Banaras Hindu University and Chairman of the Reception Committee. It was then formally inaugurated by Dr. Sampurnanand, Chief Minister, U.P., who also declared open the Mathematics Exhibition.

Secretary's Annual Report

Professor S. Mahadevan, Secretary of the Society, then presented his report for the year 1954-55. He conveyed the Society's thanks to the Banaras Hindu University for their kind invitation to hold the twenty-first Conference at Banaras.

During the year under report the Society lost five of its members: Professor S. B. Belekar, Professor G. V. Bhagawat, Rev. E. de Rafael, Dr. V. P. Venkatachari, Professor T. Vijayaraghavan and Professor H. Weyl. Professor Belekar served on the Council of the Society for a period of two years. Both Professor Bhagawat and Rev. de Rafael were esteemed members of long standing. Dr. Venkatachari as Local Secretary of the last Conference of the Society at Hyderabad was largely responsible for its success. Professor Vijayaraghavan was a member of the Council for many years and had served the Society as President, Secretary and Librarian. Professor Weyl was an honorary member of the Society. The Secretary placed on record the Society's deep sorrow on the loss of these esteemed colleagues.

The Council of the Society was reconstituted during the current year in accordance with Article VIII of the Constitution. The Secretary wished to thank those members on the old Council who

retired after a period of useful service, particularly Dr. C. N. Srinivasiengar, who was the Treasurer of the Society.

Referring to the Library of the Society the Secretary said, "Our Library is located at the Ramanujan Institute of Mathematics, Madras, and is under the care of Professor C. T. Rajagopal, Acting Director of the Institute. He is assisted by a local Committee who help him in selecting books for the Library. During the year under report, we purchased books worth nearly Rs. 3,500 and have effected also valuable exchanges with recent journals." The Secretary invited members to take increasing advantage of the facilities for the circulation of current periodicals among members.

The Secretary mentioned that both the periodicals of the Society continued to be edited by Professor K. Chandrasekharan.

The Secretary continued, " It is gratifying to mention that we will be having in Bombay, in the last week of February next, a Conference on Mathematical Education in South Asia. This will be organized by the Tata Institute of Fundamental Research with the support of UNESCO, the International Mathematical Union, and the National Committee for Mathematics in India. It is expected that a large number of foreign mathematicians will also take part in the Conference. We hope that all universities and scientific bodies will send delegations to place their views before it." The Society intended to bring out a special number of *The Mathematics Student* in connection with this Conference; for this it requested the Government of India for an increased publication grant. The Secretary added that the Tata Institute proposed to arrange an International Colloquium on Zeta Functions also in February.

The Secretary was grateful to Professor M. Eichler of the University of Muenster, West Germany, for attending the Conference and for agreeing to deliver a one-hour address.

The Secretary thanked the Government of India for having placed the Society on the permanent list of scientific bodies whose members are entitled to travel concessions. Finally, he said, " We

have to acknowledge with gratitude the help and encouragement we have been receiving from the Government of India, the Tata Institute of Fundamental Research, the National Institute of Sciences of India and the Universities of Bombay, Madras and Osmania. We appeal to the Government of India to give us help by increasing their grant and to other universities to give us grants. In conclusion, I appeal to all teachers of mathematics to join the Society in large numbers, and to those present here and others interested in mathematics to extend their hearty co-operation to the Society in its efforts to promote higher mathematical study and research."

Award of the Narasinga Rao Medal

The Narasinga Rao Medal for the previous year was then presented to Mr. B. V. Singbal for his paper entitled "Algebraic basis of Morse's variational theory", *Journal of the Indian Mathematical Society*, 18, 1954.

Address

Professor R. Vaidynathaswamy then delivered an address which is printed separately.

Business Meeting of the Society

The Council of the Society met on the afternoon of December 24. At the meeting of the General Body which followed, a resolution of condolence on the deaths of Professors Belekar, Bhagawat, Rev. de Rafael, Venkatachari, Vijayaraghavan and Weyl was passed.

Proceedings of the Conference

Over sixty papers were presented to the Conference during the two sessions devoted to the reading of papers. Abstracts of these papers appear elsewhere. On December 24, Professor M. Eichler

of the University of Muenster delivered, by special invitation, an address on 'Ramanujan's τ- function.' On December 25, there were two invited addresses, one by Professor F. C. Auluck on 'Stability problems in magneto-hydrodynamics', and the other by Dr. K. G. Ramanathan on 'Arithmetic over algebras'. A symposium on 'Unified field theory' was held on December 24, and one on 'Entire and meromorphic functions' the next day. Dr. G. S. Mahajani delivered a popular lecture on 'Extension of the universe in space and time' on December 24. On December 25 there were two popular lectures, one by Dr. P. L. Bhatnagar on 'Our Sun', and the other by Professor K. S. Krishnan on 'The geometry of our universe'.

Social Programme

The Entertainment of the delegates included receptions and a variety show. On December 26 the delegates went on an excursion to Sarnath and an excursion to the Bathing Ghats on the Ganges was arranged on the following day.

ADDRESS

By R. VAIDYANATHASWAMY

It is nearly two decades since I last had the privilege of addressing a conference of this Society. Much has happened during this period. The political and cultural situation has changed, and the conditions of the scientific work and research have improved, mainly through the progressive spirit which animates our new Central Government. Some part of the difficulties which the Society has laboured under in the past have been removed, but at the same time, there are many and formidable problems which are looming ahead of us. It is perhaps a happy augury of the future, and a sign of the altered conditions that we are meeting for the first time in this city of hoary traditions of Indian learning and Indian culture.

To get an idea of the distance covered in mathematical advancement from the founding of this Society to the present day, one has only to cast one's mind back some forty or fifty years. The original founders of the Society, V. Ramaswamy Aiyer and M. T. Naraniengar were captivated by the elegances of mathematical reasoning mostly in one branch. It was the fascination of this subject which supplied the impulsion for the founding of the Society and the starting of its journal, as well as the cement for its continued existence during the earlier phase of its career. At that time the idea of scientific research as a co-operative international effort was not known, and the necessity of distinguishing the known result and the new result, and the obligation of referring the former to its first ascertainable discoverer was not appreciated. The idea of research itself was unknown to the few universities which then existed; it was only very gradually, and through the increasing numbers of Indians trained in British and foreign universities, that the wide range of modern mathematics began to be realised, and a few modern topics and theories began to be introduced in our university curricula, and to be encouraged as subjects of study and research. At the same time our isolation has been broken to some extent in recent years

by the visits of mathematicians of the first rank to our conferences and institutions, and their talks on live subjects of research. Our recent contacts with American centres of learning, and with the International Mathematical Union have greatly strengthened the cause of mathematics in this country. The two journals of the Society are run on international standards of research and have now greatly improved at the hands of their talented editor, Prof. K. Chandrasekharan of the Tata Institute of Fundamental Research.

While we may be glad at these improvements, I need hardly say that we are far away from the ideal of making this country take a respectable place among other advanced countries, in respect of mathematical output and mathematical culture. There are practically no books of advanced level published in the country, and of the many and wide ramifications of modern mathematics, there are only a few with which the teachers here are conversant in some degree. While there is some pursuit of research in our universities, its average level is consequently, low, and it is often not well-directed. There are two research institutes concerned with mathematics in the country, the Ramanujan Institute, which has published work on analysis and topology, and the Tata Institute of Fundamental Research, in which a serious attempt seems to be made to train students in some of the more modern theories of mathematics. Apart from these, the general level is poor. There is no institution in the country, other than this Society, which can be expected to feel a specific sense of responsibility for this state of affairs. The level of mathematical achievement is an index of the general scientific and cultural level, and it is up to the Society to meet the challenge of the situation, with suitable remedial measures.

During the last fifty years mathematics has been advancing at such a rapid pace, and evolving so many fundamental ideas, and ranges of application, that post-graduate education has not had time to catch up and adjust itself. It will probably take another fifty years for the influence of these advances to settle down, and effect suitable modifications both in the graduate and the post-graduate teaching programmes. But advanced research cannot be

separated from teaching, and the task of assimilating modern developments requires the introduction of these modern theories more and more in our M.Sc., and Honours courses. At present, the Honours and M.Sc. courses in most Indian universities are old-fashioned and do not follow the spirit of modern mathematics.

Advanced mathematical research in a country can only be the dome of a well-built edifice of educational and scientific organisation. There has been a long standing complaint about the low level of our graduates, and that is certainly true of standards in mathematics. The main blame for this has been rightly laid upon indiscriminate admissions and mass education practised in our universities; it is to be hoped that with the advent of the higher secondary school, and the three-year degree courses in all universities, professional and technical courses will also become available to the bulk of the students, and only the select few will come to the universities. At the present moment, the southern universities have already decided upon the three-year degree course, even without the provision of alternative courses, and it is feared on all hands that standards will go down further with a bang. While the whole of college education is thus in the melting pot, it is difficult to prefigure the pattern of mathematical courses which will emerge, or to make any suggestions about its right relationship to the actual level of mathematical science.

In the case of standards in mathematics there is also the other reason to be mentioned, namely the abstract and formal character of its teaching. It is a trite observation that mathematics should be taught along with its applications, but I do not believe that an average college teacher of mathematics knows even a small part of the applications which significant formulae in mathematics have found in technology. It would be worth while for a well-informed committee to list all the applications which it will be desirable to refer to in college teaching. In this context must be mentioned the fact that a degree course in mathematics cannot at present be combined with a study of other sciences except at the 'subsidiary' level. It might of course be possible to change this in a three-year

degree course, if sentimental ideas about compulsory teaching of a number of languages do not come in the way.

Mathematical discovery itself being individual cannot be organized, but between the level of creative mathematics and the level of university research, there is an intermediate level of consolidation, the careful organisation of which is a *sine qua non* of scientific progress. Today, we have in the country some mathematicians who have done distinguished work; this Society can well initiate a scheme whereby they could be invited to write review-books complete in themselves on subjects of their choice and present them for publication to the Society. Naturally, they would choose subjects in which they had laboured already, and their special insights would thereby be preserved, and embodied in tradition. It is only thus that mathematical research could develop in this country, and only thus that a body of informed scientific opinion can be formed, the influence of which, it may be hoped, will gradually wean the universities away from their present position of parochial units, subject to non-academic influences.

It is customary in an inaugural address to speak on some mathematical topic of general interest. I have already mentioned the part played by geometry in the founding of this Society, and it strikes me that I could hardly do better than talk of geometry, or rather of the broad division of mathematics into two parts, geometry and analysis. This distinction is a traditional one, and is often considered to correspond to pairs of opposites current in language, such as continuous *versus* discrete, intuition *versus* logic. The Hindus are credited with the discovery of algebra (which may be considered as the first stage of analysis), while geometry is attributed to the Greeks. If the correspondence of geometry-analysis to intuition-logic is well founded, it would be difficult to explain the curious fact, that Hindu mathematics has been the more intuitive and the Greek mathematics the more logical ; witness for instance the Hindu failure to formulate the notion of limit, and Euclid's geometry being developed deductively from a postulational system. We may even go further and reverse the apparent situation, by noting that the

motivation behind the Hindu algebra was the elucidation of planetary motions, while Euclid's geometry was linked to the Pythagorean mysticism about integers and irrational numbers.

It would appear on the other hand that the distinction between geometry and analysis should be brought into relation with the two levels of our mental consciousness, one overlaid on the other, which may be roughly described as visual thought and auditory thought respectively. Auditory thought is generally considered as the usual form prevalent among civilized men, while visual thought, particularly in its pure form, unmixed with auditory sensation, is rare and apparently occurs only among certain artists and sensitive persons. But the truth is, that visual thought is really universal among men, but on account of the greater force of auditory thought, which is linked with the specifically human power of speech, it has been mostly suppressed into the unconscious. It may be asked: is there any evidence that it functions in the unconscious in all persons? The required evidence lies in the process of understanding—say, an oral communication; what happens before the moment of understanding is a lightning-flash of a visual presentation in the unconscious ; the presentation itself is not usually accessible to normal consciousness, though it is known to Yogis, and it may sometimes be possible for a person with a calm mind accustomed to introspection, to catch the trail of the presentation. It is indeed the happening of this presentation that produces the feeling of understanding in the conscious mind, and its absence, that produces the negative feeling of blankness.

The existence of the visual element behind understanding is also supported by the Tantrik theory of the speech-process. According to this theory the speech-process has four successive stages, called respectively *Para Vak*, *Pashyanti Vak*, *Madhyama Vak* and *Vaikhari Vak*, these being located in order in the *mooladhar*, the navel, the heart and the throat. Here, the *Vaikhari Vak* is the uttered word, the *Madhyama Vak* is the Vedic *Saraswathi*, and corresponds to auditory thought, the *Pashyanti Vak* is pure visual thought, while *Para Vak* is formless thought known only to great

Yogis. Just as the thinker practises, in thinking, the blocking of the speech-process at the level of *Madhyama Vak,* or auditory thought, so the Yogis by practising the silence of the mind, try to block it at the deeper level of visual thought; but this is very difficult, as the *Pashyanti* has a strong tendency to burst into word-form.

This psychological discussion would indicate that the visual element present in all understanding, would also be operative in mathematics. We may connect this visual element with the *Akasatatwa*, or spatialising power, which according to the Sankhya analysis, is one of the ultimate operative principles of existence. From this point of view, it would indeed seem, that the historically earlier definition of mathematics as the science of extension would be nearer the mark than the modern logistic definition, hailing from Russell, that mathematics is the science of necessary consequences of an abstract system of postulates. As the most abstract mathematical concept or reasoning is understood through a corresponding visual presentation in the unconscious, it follows that the task of mathematical discovery and research is related to our becoming increasingly aware of these presentations, or raising them more and more effectively to the level of our consciousness. In this connection the directions of advance of mathematics from the beginning of this century appear to be highly significant; topology, the theory of linear and function spaces, the theory of order, are all the results of our having become overtly conscious of the concealed forms of visual thought which have all along been operating in the unconscious psyche.

All this naturally leads to the question of the actual relation between number and space. This is not quite the same question as the relation between analysis and geometry; for, of course, the spaces studied in pure mathematics are by definition, images of number-systems, or, in other words, geometry in pure mathematics, has been reduced to disguised analysis. But do the abstract spaces of mathematics reproduce the actual space of our experience? Does the number-system of analysis furnish an adequate tool

to describe our actual and possible spatial experience? Euclid made integral and fractional numbers correspond to points on a line, by means of the notion of length as measured by a rigid scale; he also defined irrational lengths, or irrational numbers, in a manner essentially similar to what we do today, namely by pairs of rational numbers whose difference tends to zero. A couple of millenniums later, Dedekind gave a rigorous postulational form to the same idea, by the concept of sections of the systems of rational numbers. Now, Euclid's geometry was not the abstract geometry of modern mathematics, but was in intention, a geometry of actual space. The Euclid-Dedekind theory of number implies that a rigid scale is susceptible of infinite division, and continues to be a rigid scale throughout the division ; in other words, it is equivalent to a denial of *infinitesimals.* It was apparent to early philosophers that there would be some stage in the division, when the rigid scale disappears as such, and if it is desired to continue the representation of numbers, some other unit of unknown nature will have to be used. Hence they held to the notion of infinitesimals, or quantities incommensurable by the same units as finite magnitudes. By this denial of infinitesimals, modern mathematics has been able to evolve a number system of uniform quality, with uniform operational properties. While this furnishes an elegant background for intellectual discipline, and is of the greatest use in elucidating finite ranges of our experience, it may quite possibly mislead in the case of very large and very small magnitudes, for example, in the case of stellar and atomic dimensions.

This notion of infinitesimal, and the correlated notion of the infinite, is closely bound up with the idea of different incommensurable worlds of experience. Let me recall to your mind, that the idea of worlds of experience or ‘ lokas ’ is part of the spiritual tradition of this country, and that theories and doctrines bound up with it—such as astrology, the theory of cyclic time or yugas—continue to be standing challenges to the outlook and methodology of modern science. The unlimited conceptual extension of the process of measurement, and the resulting number system of

uniform quality leading to rigorously definable limit-processes, does not appear to mirror a situation in which there are incommensurable worlds of experience. This might have been the veritable psychological obstacle which prevented the Hindu mathematician Bhaskara from formulating the notion of limit, and the foundation of modern analysis.

It should be mentioned also that the recent mathematician and philosopher Ouspensky has rediscovered the connection between the notion of 'infinitesimal' and that of 'worlds of experience' and the instructive expositions of the idea may be found in his books.

Leaving aside these intricate questions, let us observe that while Euclid laboured to arrive geometrically at the concept of irrational number, Descartes reversed the procedure, and was able to see that the spatial relations of geometrical figures were merely the forms of algebraic and analytical relations between coordinate numbers and functions. Geometry thus became submerged in analysis, as its outward or visual form, and this welding together has never been loosened, except in certain abstract mathematical theories. In fact, analysis and the real-number system may be considered to be abstractions of the first degree, while mathematical theories which do not refer at all to the real-number system may be called abstract in the second degree. Geometrical theories which are abstract in the second degree are exemplified by general or set-topology, and the theory of foundations of abstract projective geometry, while spaces which are abstract only in the first degree are exemplified by linear spaces and function-spaces. Of course at the level of the university degree, we are concerned only with simple cases of geometry of the first degree. The difficulty of the student in learning analytic geometry for the first time, is the psychological one of having to pass to and fro between the visual form and its analytic justification and prediction. With training, this can develop into the power, which belongs to the inventor and the genius, of simultaneously thinking in the two directions. The experience of this intimate relationship between the visual form and the analysis is perhaps the first great inspiring

moment in the intellectual development of the college student of mathematics.

The invention of non-Euclidean geometry led to the systematic study of geometry on curved surfaces. The method of representation by co-ordinates could be extended to a limited region of such surfaces, and their local properties could be studied with differential equations taking the place of algebraic equations. Such surfaces could be considered to be imbedded in a flat space. These flat spaces appear in two forms, the real or complex vector space and the real or complex projective space. It is the vector-space which has applications to mechanics and physics, and is generalised to the infinite-dimentional vector-spaces of modern analysis. But it is in the study of projective space that the amazing powers of the linear transformation, or the projection-transformation are brought home to the student. In projection, the student finds the specific metrical features of a geometrical figure dissolve before his eyes, while a single descriptive result can be made to take protean metrical forms through suitable projections. There is no doubt that this is the second great inspiring moment in the development of the college student. Projective geometry, the algebra of linear transformations, of matrices and of invariants, constitute a necessary basis of equipment for the student to proceed to the study of higher mathematics; they are in close inter-connection and may profitably be studied together. Among the remarkable applications of invariants in elementary geometry must be mentioned the class of theorems called *porisms*; these state that a geometrical figure with assigned properties either does not exist at all, or an infinity of such figures exist. I believe that the porisms make a deep and lasting impression on the aesthetic feelings of the young student; it is of course a pity that not all invariants have interpretations through porisms.

The Honours courses in South Indian universities invariably include projective geometry. While there have been many workers in projective geometry in Northern India, I believe that the subject is not a favourite to the same extent in the northern universities.

The idea of transformation-group and invariant vividly introduced in projective geometry can be extended to more complicated situations; for example, the problem of studying the orbits of a dynamical system which are near a periodic orbit can be reduced by contact-transformation to that of studying the orbits near a position of equilibrium.

For geometry on an algebraic surface or curve, we have synthetic methods depending on the theory of correspondences derived from the great Italian geometry. These are closely connected with the theory of algebraic functions and their integrals and have not been studied in this country.

The theory of linear systems of algebraic curves or surfaces is nowadays tackled by representing them on a flat projective space. This is the natural extension of the method of projective geometry, and the technique of using it is an important tool in geometrical research.

A remarkable turn to projective geometry, which brings it into close relation with modern abstract algebra, is obtained by seeking to develop it axiomatically from undefined notions of point, line, plane, etc. having the same mutual relations as in ordinary projective geometry. It was discovered by Hilbert that Desargues's theorem of perspective triangles will hold in such a geometry without any further assumptions, if and only if the dimension is at least 3. On the other hand, Desargues's theorem will hold over a plane projective geometry thus defined only if coordinates for the points and lines can be introduced in the usual manner, but from an abstract field. Of course it is a far cry from this plane projective geometry over an abstract field, and the real projective geometry that we know. The gap between them is to be bridged by successive restrictive postulates, which specialise the abstract field into the real number field. The plane projective geometries in which Desargues's theorem does not hold are called non-Desarguesian. The most general of such geometries are not capable of co-ordination from any abstract algebraic system, while the less general ones can

be furnished with co-ordinates from algebraic systems more general than a field. It appears to me that this is a very good subject for post-graduate study in our universities, as it can be combined with modern algebra to the great advantage of both.

This is rather a scrappy review of geometry and its possibilities in relation to our own position in mathematics and our future work—a review natural to one who like myself has been a participant in the great geometrical enthusiasm which found expression in the founding of this Society. I would like to close this brief address by mentioning three porisms which I have contributed to the literature, and calling the attention of geometers to the fact that they merit further investigation.

The first porism is published under the title 'A postscript to Poncelet's theorem' in *Proc. Lond. Math. Soc.* The well-known Poncelet porism states that if one triangle can be inscribed in a conic S, with its sides touching respectively three other conics S_1, S_2, S_3, cutting S in the same four points, then an infinity of such triangles exist. By a slight turn in the argument leading to this theorem, it is shown, that the theorem is still true, if S_1, S_2, S_3, cut S in the same two points and touch it at different points elsewhere.

The second porism was published in *Journ. Lond. Math. Society* in the thirties, and is a curious extension to n dimensions, of a known porism relating to the twisted cubic, *viz.* there is always one tetrahedron which is inscribed in a twisted cubic and circumscribed to a quadric surface, but if there is more than one, there must be 2 such tetrahedra. The extension to n dimensions replaces the twisted cubic by the rational norm curve and is on similar lines. Curiously enough the extension depends on the elementary fact that the rank of a skew-symmetric determinant is an even number. The proof is by means of a linear correspondence between the quadrics of n-space with the linear complexes of $(n+1)$-space; that the two families are equi-dimensional is seen from the fact that the number of independent terms in a symmetric matrix of order n and a skew symmetric matrix of order $n+1$ is the same.

The third porsim (A fundamental property of Hart systems oi circles, *Proc. Ind. Academy of Sciences*) is intimately related to plane geometry. It is well known that the four circles inscribed in a triangle are all touched by four other circles, namely the sides of the triangle and the nine-point circle. The porism states in effect that 8 circles belongs to a cubic family which contains an infinity of pairs of four circles related in the same way. For the investigation of this porism we have to use a representation of the isogonal point-pairs of a triangle by points on a four-nodal cubic surface in 3-space.

ABSTRACTS

ALGEBRA

K. G. RAMANATHAN, Bombay : *Units in p-adic algebras*

Let k be a p-adic number field and o its set of integers. Let D be a division algebra of finite rank over k and $\mathscr{V}$ the maximal order in D relative to o. Let $\mathscr{M}_n$ be the algebra of n rowed square matrices over D. An element M of $\mathscr{M}_n$ is said to be a unit if M and M^{-1} have all their elements in $\mathscr{V}$. The units of $\mathscr{M}$ form a group Γ. We first give a simple proof of Schilling's theorem that Γ has a finite set of generators in the sense that every element of Γ can be expressed as an infinite convergent product of units which are finite products of a finite number of units—the convergence is in the sense of p-adic topology.

Let D have an involution and let this involution be extended to $\mathscr{M}$. Let S be a matrix in $\mathscr{M}$ which is either symmetric or skew symmetric under this involution. A unit U of $\mathscr{M}$ is said to be a unit of S if $S = S[U]$. The units of S form a group $\Gamma(S)$ which also is shown to be finitely generated in the above sense.

Finally we give expressions for local measures of representation of quadratic forms over D in terms of measures of certain topological groups.

T. VENKATARAYUDU and T. S. G. KRISHNAMURTY, Waltair : *Representations of a finite O_h^7 space group*

The finite space group G obtained from O_h^7 by taking twice the primitive translations as equivalent to the identity element is of order 384. This group G has twenty non-equivalent irreducible representations. In this paper, a set of matrices which define these irreducible representations is given.

K. SAVITHRI, Bombay : *Number fields and function fields*

In this paper some relationships between quadratic fields over the rational number fields and quadratic fields over the field of rational functions over a finite field of constants are pointed out.

V. S. KRISHNAN, Madras : *Theory of d-structures*

This study is similar in nature to another on '*c*-structures' presented earlier, and published under the title 'Closure operations on *c*-structures'. The main point of difference is that the 'direct product' and the 'substructure' of the earlier theory are here replaced by the 'direct sum' and the 'quotient structure', and 'lattice products' by 'lattice sums'.

As before the theory is axiomatic, and there are four basic concepts; the 'd-structures', and the 'd-homomorphisms' between them, the 'a-structures', and the 'a-homomorphisms' between them. Besides the d-structures having bases which are a-structures, there are a-structures also over demi-groups with zero. The analogues, and in a sense complementary forms, of the seven axioms of the earlier theory are supplemented by an eighth axiom which corresponds to a parallel result that was provable in the earlier theory. The axiom which asserts that, given a d-homomorphism f of a d-structure D in another D' there can be found a finest comparable d-structure to D' for which f is a d-homomorphism from D, ensures that the quotient structure or the direct sum is unique, when defined.

Semi-groups, or groups, with a topology, semi-uniformity or partial order, that is, in a sense homogeneous w.r.t. the group structure, provide examples of these d-structures. Closure operations are also discussed as in the earlier theory, and some of their interrelations studied.

M. VENKATARAMAN, Madras : *On ordered abelian groups*

It is shown that every ordered abelian group whose convex subgroups (those which contain with any two elements every element lying between them) satisfy the ascending chain condition, is a subgroup of the direct product of a well ordered family of real number groups lexicographically ordered.

The question of the 'metrizability' of a topological space by a 'metric' whose values are in such an ordered group (instead of the real numbers) is also considered.

M. S. Huzurbazar, Bombay : *The eigenvalues of matrices with real quaternion elements*

In this paper we consider matrices with real quaternion elements and their eigenvalues. A real quaternion λ is called an eigenvalue of a square matrix A if there exists a non-zero column vector with $AX = X\lambda$. Using the regular representation by real matrices we prove:

(i) Every square matrix A has an eigenvalue, and the eigenvalues of A are all real quaternions similar to the eigenvalues of the corresponding matrix in the regular representation.

(ii) If λ is an eigenvalue of A then there exists a non-zero row vector Y with $YA = \lambda Y$ and conversly.

We also generalize a result due to S. N. Roy [*Proc. Amer. Math. Soc.*, 1954] and obtain bounds for the eigenvalues of a product of matrices (not necessarily square).

Nisar A. Khan, Aligarh : *Limits for the characteristic roots of the product of matrices*

Let $S_i(A)$, $(T_i(A))$ denote the sum of the absolute values of the elements in the i-th row (column), of a real or complex square matrix A; $S(A)$, $(T(A))$ the greatest of the $S_i(A)$, $(T_i(A))$; $c(A)$ a characteristic root of A ; and $\bar{c}(A)$ its conjugate. Then we establish

Theorem I. *For any characteristic root $c(AB)$ of AB, we have*

$$|c(AB)| \leqslant \frac{S(A)+T(A)}{2}\cdot\frac{S(B)+T(B)}{2}.$$

Equality holds when $A = B = I$.

If we put $B = I$ in the above result, we get the result of E. T. Browne [*Bull. Amer. Math. Soc.*, 36 (1930)].

If corresponding to the square matrices $A = (a_{ij})$ and $B = (b_{ij})$ we form two real symmetric matrices $\alpha = (\alpha_{ij})$ and $\beta = (\beta_{ij})$ respectively, such that $\alpha_{ij} = \frac{1}{2}(|a_{ij}| + |a_{ji}|)$ and $\beta_{ji} = \frac{1}{2}(|b_{ij}| + |b_{ji}|)$ and the greatest values of $\sum_{s=1}^{n} \alpha_{rs}$ and $\sum_{s=1}^{n} \beta_{rs}$, (for $r = 1, 2, \ldots n$), are denoted respectively by $S'(A)$ and $S'(B)$, then we prove:

THEOREM 2. *For any characteristic root $c(AB)$ of AB, we have*

$$|c(AB)| \leqslant \{S'(A)\} \{S'(B)\}.$$

P. S. RAU, Tirupathi : *On positive definite quadratic forms*

We prove from elementary determinant theory the well-known result, viz. a necessary and sufficient condition that a homogeneous quadratic function in n-variables may represent a positive definite form is that its discriminant and the principal cofactors of every order of its discriminant are positive.

The result is verified in the case of one, two and three variables and the general case is deduced by the method of induction. The method employed in deducing the case of three variables from that of two variables leads to generalization by using the following lemma, from determinant theory, viz.

$$\mathcal{M}\begin{pmatrix} r_1, r_2, \ldots r_p \\ i_1, i_2, \ldots i_p \end{pmatrix} = (-1)^{\Sigma(i_t+r_t)} M^c \begin{pmatrix} r_1, r_2, \ldots r_p \\ i_1, i_2, \ldots i_p \end{pmatrix} \Delta^{p-1},$$

where Δ is a determinant of order n, $\mathcal{M}\begin{pmatrix} r_1, \ldots r_p \\ i_1, \ldots i_p \end{pmatrix}$ is a minor of order p formed from the constituents of $i_1, i_2, \ldots i_p$ rows and $r_1, r_2, \ldots r_p$ columns of the reciprocal of Δ, and $M^c \begin{pmatrix} r_1, \ldots r_p \\ i_1, \ldots i_p \end{pmatrix}$ is the complementary of the minor of order p formed from the constituents of the $i_1, i_2, \ldots i_p$ rows and $r_1, r_2, \ldots r_p$ columns of Δ.

M. V. SUBBA RAO, Tirupathi : *On a function connected with $\phi(n)$*

The late Dr. S. S. Pillai has considered (*Bull. Amer. Math. Soc.* 1929) the function $R(n)$, defined as follows. Let $\phi_1(n) = \phi(n)$, the number of numbers less than and prime to n; $\phi_2(n) = \phi[\phi_1(n)]; \ldots$. Then $R(n)$ is defined to be the least value of t for which $\phi_t(n) = 1$. I consider the function $S(n)$, given by $S(n) = \phi_1(n) + \phi_2(n) + \ldots + \phi_t(n)$, $t = R(n)$. It is shown here that $S(n) \leqslant n - 1$ (n even), $\leqslant 2n - 3$ (n odd), and $\geqslant [\log(n/2)/\log 3] + 1$. Also we show that

$$R(n) \leqslant \frac{\log n}{\log 2} \ (n \text{ even}), \leqslant \frac{\log(n-1)}{\log 2} + 1 \ (n \text{ odd}),$$

which are slight improvements over Pillai's results. I also consider the solutions of the equation $S(n) = n$; and it is shown that the

following are among the solutions : $n_1 = 3^k$, $k = 0, 1, 2, \ldots$; $3n_2$, $3n_3$, $3n_4, \ldots$ where $n_2, n_3, n_4, \ldots$ are primes given by $n_2 = 1 + 4n_1$; $n_3 = 1 + 12n_2$; $n_4 = 1 + 12n_3$; ... It is conjectured that these are the only solutions.

A. C. CHOWDHURY, Calcutta : *On trioperational systems*

NISAR A. KHAN, Aligarh : *The characteristic roots of the product of two matrices*

C. R. MARATHE, Aligarh : *Properties of μ-matrices*

C. R. MARATHE, Aligarh : *On quasi-idempotent matrices*

C. R. MARATHE, Aligarh : *A matric equation involving the auxiliary unit matrix*

P. V. KRISHNIAH, Waltair : *On Leudesdorf's congruences*

D. R. KAPREKAR, Deolali : *A particular constant of a series of reversed differences*

K. SATYANARAYANA, Rajhamundry : *Dimensional check of formulae*

ANALYSIS

M. V. SUBBA RAO, Tirupathi : *Closure theorems on a certain space of sequences*

Let S_p denote the set of all sequences $x = (a_1, a_2, \ldots)$ such that $\Sigma |a_n|^p$ is convergent $(p > 1)$. This set can be normed, the norm being

$$|x|_p = (\Sigma\, |a_n|^p)^{1/p}.$$

Then we get the space usually denoted by l_p. Now let us impose on the set S_p the topologies defined by the norms of l_q, for all $q > p$. Let l_{p+} denote the lattice product topology. Then l_{p+} is stronger

than l_p topologically. The topology l_{p+} is metrizable with distance function $d(x, y)$,

$$d(x, y) = \sum \frac{1}{2^n} \frac{|x - y|_{p+1/n}}{(1 + |x - y|_{p+1/n})},$$

and l_{p+} is a complete space in this metric. The functionals of this space are studied ; and the following two closure theorems are proved.

THEOREM 1. *Let* $\alpha = (a_1, a_2 \ldots)$ *be an element of* l_{p+} $(a_n \neq 0)$. *Let* $\alpha_n = (a_1 z_n, a_2 z_n^2, \ldots, a_i z_n^i, \ldots)$, $n = 1, 2, \ldots$, *be a sequence of elements of* l_{p+}, *where* (z_n) *is a sequence of complex numbers such that* $|z_n| \leqslant 1$ *and either* (i) *having a limit point in the unit circle or* (ii) $\Sigma(1 - |z_n|)$ *divergent. Then the closure of the linear manifold spanned by* (α_n) *is* l_{p+}.

THEOREM 2. *Let* $\alpha = (a_1, a_2, \ldots) \in l_{p+}$ *be such that* $\mathrm{Rl}(a_n)$ *is a strictly decreasing sequence. Then the closure of the linear manifold spanned by* (α_n) *where* $\alpha_n = (a_1^n, a_2^n, \ldots)$, $n = 1, 2, \ldots$, *is the whole space* l_{p+}.

T. PATI, Allahabad : *On the absolute Cesàro summability of Fourier series generated by functions of Lebesgue class* L^p *and some related theorems in the theory of Fourier constants*

We write $\mathscr{S}[f]$ to denote the Fourier series of $f(t)$ which is assumed to be periodic with period 2π and integrable in the sense of Lebesgue. Further, throughout the sequel, k is some constant $> \pi$, $q = p/(p-1)$, where $p > 1$, $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$, and $\phi(t) \sim \Sigma a_n \cos nt$.

The question whether summability $|C, \delta|$, $0 < \delta \leqslant 1$, at a point, of $\mathscr{S}[f]$, where $f \in L$, is a local property of $f(t)$ was answered in the negative by Bosanquet and Kestelman (1939). In 1940 Alberto Foà demonstrated with the help of an example that for $\mathscr{S}[f]$, $f \in L^p$, $p > 1$, summability $|C, 1/p|$ at a point is not a local property. The first theorem showing that summability $|C, \delta|$, $\delta > 1/p$, of $\mathscr{S}[f]$, $f \in L^p$, $1 < p \leqslant 2$, is a local property is due to Sunouchi and Tsuchikura (1949). The most refined result in this

direction is that of Tsuchikura (1953 & 1954): If $f \in L^p$, $1 < p \leqslant 2$, and $\int_0^\pi \{|\phi|^p/t\} (\log k/t)^\alpha \, dt < \infty$, for some $\alpha > p - 1$, then $\mathscr{S}[f]$, at $t = x$, is summable $|C, \delta|$ for every $\delta > 1/p$. In the first part of the paper, by making use of Tsuchikura's result, the author establishes the following theorem.

THEOREM 1. *If* $1 < p \leqslant 2$, $f \in L^p(0, 2\pi)$ *and* $\int_0^\pi \{|\phi|^p/t\} \times \log (k/t)^\alpha \, dt < \infty$, *for some* $\alpha > p - 1$, *then* $\Sigma n^{-\delta} \, |\int_0^\pi \{\phi(t)/(\sin t/2)^\delta\} \times \sin \{(n + \delta/2) t - \frac{1}{2}(\delta - 1)\pi\} \, dt| < \infty$, *where* $\delta > 1/p$.

In the second part of the paper the author discusses certain problems concerning Fourier constants. It is trivial that if $\phi \in L^p(0, \pi)$, $1 < p \leqslant 2$, then $\Sigma n^{-\delta} \, |a_n| < \infty$, for every $\delta > 1/p$. It can be shown by appealing to a result of Hardy and Littlewood that if $\phi(t) \, t^{\delta - 1/p} \in L^p(0, \pi)$, $1 < p \leqslant 2$, then $\Sigma (\log n)^{-\epsilon} \, n^{-\delta} \, |a_n| < \infty$, where $\delta \geqslant 1/p$, for every $\epsilon > \delta$. On the basis of his Theorem 1 and the non-local nature of summability $|C, 1/p|$, $1 \leqslant p \leqslant 2$, of $\mathscr{S}[f]$, $f \in L^p$, the author proves

THEOREM 2. *If* $1 < p \leqslant 2$, *and* $\phi(t) \, (\log k/t)^k \in L^p(0, \pi)$ *for some* $k > 1/q$, *it is not necessarily true that* $\Sigma n^{-1/p} \, |a_n| < \infty$.

He finally raises the problem whether, if $1 < p \leqslant 2$, and $\phi(t) \, t^{\delta - 1/p} (\log k/t)^k \in L^p(0, \pi)$, for some $k > 1/q$, then $\Sigma n^{-\delta} \, |a_n| < \infty$, for $\delta > 1/p$.

SULAXANA KUMARI, Allahabad: *On the non-convergence of Fourier series*

Let $f(\theta)$ be a function integrable in the sense of Lebesgue over $(-\pi, \pi)$ and periodic outside this range, with period 2π. Let the Fourier series associated with $f(\theta)$ be $f(\theta) \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = A_0/2 + \sum_{n=1}^{\infty} A_n(\theta)$. Let $\phi(t) = \frac{1}{2}\{f(x + t) + f(x - t) - 2s\}$.

It has been proved by Wang [*Proc. London Math. Soc.*, 47 (1942)] that if, for $k < 1$, (i) $\int_0^t \phi(u) \, du = o(t^k)$, as $t \to 0$, and if (ii) $A_n(x) > -K \, n^{-1/k}$, then the Fourier series converges to the sum

s at $\theta = x$. He has shown by means of an example that k in condition (i) cannot be replaced by any $k' < k$.

In the present paper the above result of Wang is shown to be best possible in the sense that even 'o' in condition (i) cannot be replaced by 'O'.

M. S. RAMANUJAN, Madras: *On series-to-series quasi-Hausdorff transformations*

In a recent paper [*J. Indian Math. Soc.* 17 (1953), 47-53], I obtained the necessary and sufficient conditions for a quasi-Hausdorff series-to-series transformation to be (i) convergence preserving, i.e. for the quasi-Hausdorff matrix (H^*, μ) to be a δ-matrix, and (ii) to be limit preserving, i.e. (H^*, μ) to be an α-matrix. Defining a δ-matrix and an α-matrix in which each row tends to 0 as a δ_0-matrix and an α_0-matrix respectively, and making use of a theorem of Vermes (*Acta Szeged*, 14 (1951) 23-38, Theorem 3.VI), I prove the following theorem for the class β of series with bounded partial sums.

THEOREM. *Let μ_n be a semi-regular moment constant, i.e. μ_n is a moment constant and $\chi(+0) = 0$. Let μ_n'/μ_n be a regular moment constant. Then for the class β, $(H^*, \mu') \supset (H^*, \mu)$.*

The above theorem is used to prove an inclusion theorem for the Hardy-Littlewood circle method of summation.

I also prove the following Mercerian theorem.

THEOREM. *Let Σa_n belong to the class β. Let $\alpha > 0$ and*

$$b_n = \alpha a_n + (1-\alpha)\left[\frac{a_n}{n+1} + \frac{a_{n+1}}{n+2} + \ldots\right]$$

Then the convergence of Σb_n implies that of Σa_n to the same sum.

Appealing to the conditions of Knopp and Lorentz (*Archiv der Math.* 2, (1949), 10-16) for a series-to-series transformation to be absolutely conservative and absolutely regular, I prove the

THEOREM. *The quasi-Hausdorff series-to-series transformation is*

(i) *absolutely conservative if and only if it is conservative;*

(ii) *absolutely regular if and only of it is regular.*

M. S. RAMANUJAN, Madras: *Some properties of double sequence transformations*

Let (d) denote the space of double sequences, in which each row, each column, and the double sequence itself, all converge to the same limit; if the limit is zero the space is denoted by (d_0). The transformation of a double sequence by a matrix $A = (a_{mnpq})$ is said to be completely conservative for (d), if every double sequence in (d) is transformed into another in (d) itself. If in addition the limit is also preserved, then the transformation is said to be completely regular for (d). The necessary and sufficient conditions that (i) A is completely conservative for (d) and (ii) A is completely regular for (d) are obtained, and the conservative matrices are proved to form a Banach algebra under the norm

$$\underset{0<m,n<\infty}{\text{l.u.b.}} \sum_{p,q=0}^{\infty} |a_{mnpq}|.$$

The following theorems are also proved.

THEOREM. *Let* $A = (a_{mnpq})$ *be completely conservative for* (d) *and let its elements satisfy the condition*

$$\chi(A) = \lim_{m,n} \sum_{p,q=0}^{\infty} a_{mnpq} - \sum_{p,q=0}^{\infty} \lim_{m,n} a_{mnpq} \neq 0,$$

and let $y = (y_{mn}) \in (d)$ *be such that* $\sum_{i,j=0}^{\infty} \alpha_{ij}\, y_{ij} = 0$, *whenever* $\sum_{ij=0}^{\infty} |\alpha_{ij}| < \infty$ *and* $\sum_{i,j=0}^{\infty} a_{ij}\, a_{ijpq} = 0$ *for* $p,q = 0, 1, 2, 3, \ldots$ *Then for every* $\epsilon > 0$, *there exists an* $x = (x_{mn})$ *such that*

$$|A_{ij}(x) - y_{ij}| \leqslant \epsilon, \quad \text{for } i,j = 0, 1, 2, \ldots.$$

THEOREM. *Let A be completely conservative for (d) and let $y \in (d)$ be such that for every $\epsilon > 0$ there exists an $x \in (d)$ such that $| A_{ij}(x) - y_{ij} | \leqslant \epsilon$ for all $i, j = 0, 1, 2, \ldots$ Then for any (α_{ij}) such that*

$$\sum_{i,j=0}^{\infty} |\alpha_{ij}| < \infty \text{ and } \Sigma\, \alpha_{ij}\, a_{ijpq} = 0, \quad p, q = 0, 1, 2, \ldots,$$

we have $\Sigma\, \alpha_{ij}\, y_{ij} = 0$.

THEOREM. *The four dimensional Hausdorff matrix is completely regular for (d) if and only if μ_{mn} is a regular moment constant.*

An inclusion condition for Hausdorff transformations is also proved. The following theorem follows from the inclusion theorem.

THEOREM. *Let (s_{mn}) be a double sequence and let $\alpha > 0$, $\beta > 0$. Let*

$$t_{pq} = \alpha\, \beta\, s_{pq} + \frac{\alpha\,(1-\beta)}{q+1} \sum_{j=1}^{q} s_{pj} + \frac{\beta\,(1-\alpha)}{p+1} \sum_{i=1}^{p} s_{iq} + \\ + \frac{(1-\alpha)\,(1-\beta)}{(p+1)\,(q+1)} \sum_{i,j=0}^{p,q} s_{ij}.$$

Then if $(t_{pq}) \in (d)$ with limit l, so does (s_{pq}).

PRAMILA SRIVASTAVA, Allahabad : *A theorem on strong summability*

Let $\sum_{1}^{\infty} a_n$ be a given infinite series, and λ_n a positive monotonic increasing sequence tending to infinity with n. We write

$$A_\lambda(\omega) = A_\lambda^0(\omega) = \sum_{\lambda_n < \omega} a_n$$

and

$$A_\lambda^r(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^r a_n, \; r > -1;$$

and the series Σa_n is said to be summable (R) to sum s, if

$$A_\lambda^r(\omega)/\omega^r \rightarrow s$$

as $\omega \rightarrow \infty$ for some $r \geqslant 0$. It is said to be strongly summable (R, λ, r) or summable $[R, \lambda, r]$, to sum s, if $\int^{\omega} | A_\lambda^{r-1}(x)/x^{r-1} - s \mid dx = o(\omega)$

as $\omega \to \infty$. If $\int^{\omega} | A_{\lambda}^{r-1}(x)/x^{r-1} | dx = O(\omega)$ as $\omega \to \infty$, the series is said to be bounded $[R, \lambda, r]$. If Σa_n be convergent and $\sum_{\lambda n < \omega} | a_n \lambda_n | = o(\omega)$, the series is defined to be summable $[R, \lambda, 0]$; and if Σa_n be bounded and $\sum_{\lambda_n < \omega} |a_n a_n \lambda_n| = O(\omega)$, the series is defined to be bounded $[R, \lambda, 0]$.

The first theorem of consistency for strong Rieszian summability states that if an infinite series Σa_n is summable $[R, \lambda, r]$, $r \geqslant 0$, then it is summable $[R, \lambda, k]$ for $k > r$. It is known that a series which is summable $[R, \lambda, k]$ is not necessarily summable $[R, \lambda, r]$, $r < k$. A result has already been obtained in the direction of determining some extra condition or conditions which together with summability $[R, \lambda, k]$ may yield summability $[R, \lambda, r]$. The object of the present paper is to establish the following theorem in connection with the same problem.

THEOREM. *If Σa_n is summable (R) and bounded $[R, \lambda, k]$, then it is summable $[R, \lambda, k + \delta]$, $\delta > 0$, $k \geqslant 0$.*

M. R. PARAMESWARAN, Madras : *On the constants associated with a reversible summability matrix*

Let A be a reversible matrix. Then the equations $y_n = \sum_{k=0}^{\infty} a_{nk} x_k$, $n = 0, 1, 2, \ldots$, have a unique solution given by $x_k = c_k \lim_{n \to \infty} y_n + \sum_{p=0}^{\infty} b_{kn} y_n$, $k = 0, 1, 2, \ldots$, where the constants c_k and b_{kn} depend only on A. Banach's statement (see Banach : *Theorie des operations lineaires*, p. 50) that $\{ c_k \}$ is a bounded sequence was shown to be false by Macphail (*Proc. Amer. Math. Soc.*, 5 (1954) 120-121). The present paper discusses some properties of the constants c_k and b_{kn}. Let $\| B \| = \text{l.u.b.}_{0 \leqslant k < \infty} \sum_{n=0}^{\infty} | b_{kn} |$. Then

THEOREM 1. *The reversible matrix A sums an unbounded sequence if and only if either $\| B \| = \infty$ or $\{ c_k \}$ is unbounded.*

THEOREM 2. *If A is a reversible K-matrix and $\| B \| < \infty$ then $c_k = 0$ for all k, and A is of type M.*

THEOREM 3. *There exist reversible (necessarily not K-) matrices A, summing an unbounded sequence and for which* $\| B \| < \infty$.

THEOREM 4. *There exist reversible K-matrices A with* $\| B \| = \infty$ *and satisfying any one of the conditions:*

(*a*) $c_k = 0$, *for all k,*

(*b*) $|c_k| < \infty$, $c_k \neq 0$ *for any k,*

(c) $c_k \to \infty$,

(*d*) *A is of type M.*

Theorem 1 is an extension of a theorem due to A. Wilansky.

M. R. PARAMESWARAN, Madras : *A note on Tauber's converse of Abel's theorem*

Let $\{H_n^\alpha s\}$, $\{C_n^\alpha s\}$ denote the sequences of Hölder and Cesàro transforms respectively (of order $\alpha \geqslant 0$) of the real sequence $s = \{s_n\}$. A classical result due to Tauber may be written in the form: *if* (i) $s_n - H_n^1 s \to 0$ (*as* $n \to \infty$), *and* (ii) $\{s_n\}$ *is Abel-summable to l, then* $s_n \to l$. The question is raised here as to what happens when we take instead of (i) above, a condition involving $s_n - k H_n^1 s$. We have :

THEOREM 1. *If* (i) *for some* $k > 1$, $s_n - k H_n^1 s \to (1-k)\, l$, (ii) $\{s_n\}$ *is Abel-summable to l and* (iii) $\{H_n^\alpha s\}$ *is bounded on one side for some* $\alpha > 0$, *then* $s_n \to l$.

THEOREM 2. *If* (i) *for some* $k > 1$, $s_n - k H_n^1 s \to (1-k)\, l$ *and* (ii) $\{s_n\}$ *is summable* (C, ∞) *to l (in the sense that* $\lim\limits_{\alpha\to\infty} \limsup\limits_{n\to\infty} C_n^\alpha s = \lim\limits_{\alpha\to\infty} \liminf\limits_{n\to\infty} C_n^\alpha s = l$) *then* $s_n \to l$.

THEOREM 3. *If in Theorem* 1, *the condition* (iii) *is dropped, then either* $\lim\limits_{\alpha\to\infty} \overline{\lim\limits_{n\to\infty}} H_n^\alpha s = \infty$ *and* $\lim\limits_{\alpha\to\infty} \underline{\lim\limits_{n\to\infty}} H_n^\alpha s = -\infty$ *or* $s_n \to l$.

M. R. PARAMESWARAN, Madras : *On the reciprocal of a K-matrix*

The set of K-matrices forms a subalgebra of the Banach algebra of K_r-matrices under the usual norm $\| A \| = \text{l.u.b.}_{0 \leqslant n < \infty} \sum_{k=1}^{\infty} |a_{nk}|$. In this

paper I prove : *If A is a K-matrix with a two-sided K_r-reciprocal A^{-1}, then A^{-1} is a K-matrix.*

S. M. SHAH, Aligarh : *The set on which an entire function is small*

We consider the points z which satisfy the inequality

$$|f(z)| \leqslant \{M(|z|)\}^{1-\lambda}.$$

Let $2\pi\, r\, l\,(\lambda,\, r)$ denote the length of that part of the circumference $|z| = r$, and E_λ the set of points z in the plane, satisfying this inequality. Let $D_R\,(E_\lambda^{\cdot})$ be the ratio $m(z \in E_\lambda,\ R_0 \leqslant |z| \leqslant R)/\pi R^2$, where R_0 is so chosen that $M(r) > 1$ for $r > R_0$, and $\bar{D}(E_\lambda^{\cdot})$ $(= \bar{D}(E_\lambda))$ and $\underline{D}(E_\lambda^{\cdot})$ $(= \underline{D}(E_\lambda))$ the upper and lower densities of $E_\lambda^{\cdot}$ (Boas, Buck and Erdos, *Amer. J. Math.* LXX, 1948, 400-402; Boas, *Entire functions*, p. 27). We prove

THEOREM 1.

$$\int_0^\infty D_R(E_\lambda^{\cdot})\, d\lambda = 1 + h - \frac{2}{R^2}\int_{R_0}^{R} \frac{r\, N(r,\, 0)}{\log M(r)}\, dr,$$

where $h = h(R,\, R_0,\, f) \to 0$ *as* $R \to \infty$.

COROLLARY. *If* $\log M(r) \sim N(r,\, 0)$ *then* $\bar{D}(E_\lambda^{\cdot}) = 0$ *for all* $\lambda > 0$.

This hypothesis is satisfied by functions satisfying the condition $\log M(r) = O((\log r)^2)$, by Weierstrass function $\sigma(z)$ etc.

THEOREM 2. *For all* $r > 0$,

$$l(\lambda,\, r) \leqslant \frac{1}{\lambda}\left\{1 - \frac{N(r,\, 0) + \log|c|}{\log M(r)}\right\},$$

where $c = c(f)$ *is a constant.*

S. M. SHAH and S. K. SINGH, Aligarh : *The defect values of a meromorphic function*

In this paper we assume that $f(z)$ is a meromorphic function of finite order and $(a_\nu)_1^\infty$, are all finite constants.

We prove

$$\liminf_{r\to\infty} T(r, f')/T(r, f) \geqslant \Sigma_1^\infty\ \delta(a_\nu)/\Delta(f',\, 0), \tag{1}$$

$$\delta(f', 0) \geqslant 1 + \Delta(f', 0) - \{\Delta(f', 0)/\Sigma_1^\infty \delta(a_i)\}\{2 - \mu(\infty) - \delta(\infty)\}. \quad (2)$$

COROLLARY. *If the maximum defect is attained at finite points, then* $\mu(\infty) = 0$, $\delta(f', 0) = 1$, $T(r, f') \sim 2\, T(r, f)$ *and* $\delta(f', \infty) = 0$.

S. K. SINGH, Aligarh : *On the exceptional values of entire and meromorphic functions*

In this paper we prove two theorems :

THEOREM 1. *Let* $f(z)$ *be an entire function of order* ρ $(0 < \rho < \infty)$ *such that* (i) $T(r, f)/r^\rho L(r) \sim 1$, (ii) $n(r, x_1)/r^\rho L(r) \to 0$. *Then* $n(r, x)/r_\rho L(r) \sim \rho$ *for all* $x \neq x_1$, *where* $T(r, f)$ *denotes the Nevanlinna characteristic function and* $L(r) > 0$, *continuous for* $r > r_0$ *and* $L(cr) \sim L(r)$ *as* $r \to \infty$ *for every constant* $c > 0$.

THEOREM 2. *Let* $F(z)$ *be a meromorphic function of order* ρ $(0 < \rho < \infty)$ *such that*

$$I = \int_{r_0}^{\infty} \frac{n(t, a)}{t^{\rho+1}}\, dt$$

is convergent for three different values of a $(0 \leqslant |a| \leqslant \infty)$; *then* I *is convergent for all values of* a *and further* $F(z)$ *is of minimal type convergence class.*

HARI SHANKAR, Aligarh : *Generalization of Lindel¨f theorems on entire functions*

E. Lindelöf has proved three theorems on entire functions of finite order for which $n(r)$ and $\Sigma Z_n^{-\rho}$ satisfy certain conditions. In this paper we extend these theorems.

Firstly we define the "L-type" of an entire function $f(z)$ of order ρ $(0 < \rho < \infty)$ as

$$\limsup_{r\to\infty} r^{-\rho} (L(r))^{-1} (\log M(r)) = \tau_L \ (0 \leqslant \tau_L \leqslant \infty).$$

Here $L(r)$ is the Karamata slowly changing function. According as $\tau_L = \infty$, $0 < \tau_L < \infty$, or $\tau_L = 0$, $f(z)$ will be called maximum, mean or minimum "L-type" of order ρ respectively.

THEOREM A. *The entire function $f(z)$ of non-integral order ρ is of mean "L-type" if and only if $n(r) = O(r^\rho L(r))$, and of minimum "L-type" if and only if $n(r) = o(r^\rho L(r))$.*

THEOREM B. *The entire function of positive integral order ρ is of mean "L-type" if and only if $n(r) = O(r^\rho L(r))$ and $\sum_1^N Z_n^{-\rho} = O(r^\rho L(r))$, where $r_N \leq r < r_{N+1}$.*

THEOREM C. *The entire function $f(z)$ of positive integral order ρ is of minimum "L-type" if and only if (1) $P = \rho$, $n(r) = o(r^\rho, L(r))$ and $C_\rho + \rho^{-1} \sum_1^\infty Z_n^{-\rho} = 0$, or (2) $P = \rho - 1$ and $C_\rho = 0$, where C_ρ is the coefficient of Z^ρ in $Q(z)$ in the Hadamard factorization of $f(z)$, and P the genus of the function.*

K. PADMAVALLY, Madras : *On a non-linear integral equation arising in connection with the heating of a solid by gas*

HARI SHANKAR, Aligarh : *A note on 'a-points' of a meromorphic function*

HARI SHANKAR, Aligarh : *Some theorems on meromorphic functions*

HARI SHANKAR, Aligarh : *A note on the characteristic function of a meromorphic function*

C. RAMAMOHANA RAO, Waltair : *On differentiation under integral sign*

S. K. LAKSHMANA RAO, Bangalore : *On a series of products of three Gegenbauer polynomials*

C. S. VENKATARAMAN, Trichur : *On the nature of θ in the Lagrange remainder in the Taylor series*

APPLIED MATHEMATICS

I. M. PANDYA and P. C. VAIDYA, Ahmedabad : *Electromagnetic fields of rotatig spheres*

A spherical radiating star is assumed to be rotating with angular velocity ω. The field outside such a star is found to be described by the line-element

$$ds^2 = -r^2(d\theta^2 + \sin^2 d\phi^2) + (1 - \frac{2m(t)}{r} + \frac{\omega f(\theta,t)}{r}dt^2 + 2drdt - \\ - 2\omega r^2 \sin^2(dr + dt)\, d\phi,$$

to the first power of ω, m and f being undetermined functions of their arguments. The line-element reduces to that of a radiating sphere when ω vanishes and it gives flat space-time at large distances. The co-ordinates are such that the wave fronts of emitted radiation are given by the family of null surfaces t = constant.

The energy tensor in the space outside the star is generated by an electromagnetic field tensor $F^{\mu\nu}$ with only F^{13}, F^{23} and F^{34} as nonvanishing components. The rotation produces a current vector in the field whose space-components, in the usual quasi-cartesian co-ordinates, are given by $J_x = -yJ$, $J_y = xJ$, $J_z = 0$, $J = O(1/r^4)$.

K. B. SHAH and P. C. VAIDYA, Ahmedabad : *Does the expanding space influence gravitational fields surrounding individual stars ?*

By fitting up the Schwarzschild's field with the expanding space of cosmology, Einstein and Strauss (*Rev. Mod. Phys.* 17, 120, 1945) have shown that " the expansion of space has no influence on the structure of field surrounding an individual star". In this paper the same problem is studied when the Einstein-Strauss singularity at the origin is a source (or a sink) of radiation. For the field round this radiating star, $T_{\mu\nu} \neq 0$, but $T_{\mu\nu} = \sigma w_\mu w_\nu$, $w_\mu w^\mu = 0$, σ being the density of flowing radiation. The differential equations for this field are solved in a manner similar to the one employed by Einstein and Strauss in their problem, but a fundamental difference in the treatment arises when one tries to fit up

this radiation field with the pressure-free cosmic field over a boundary $r = R_1(t)$. The necessary boundary conditions are studied at $r = R_1(t)$ in the light of recent work by O'Brien and Synge (*Commun. Dublin Inst. Adv. Stud* A, No. 9, 1952). It is found that one of these necessary conditions, viz. the continuity of T_k^4 (evaluated in an appropriate co-ordinate system) cannbt be satisfied at $r = R_1^o(t)$. The conclusion is therefore drawn that when a star radiates energy one *cannot* locate a definite boundary $r = R_1(t)$, such that on one side of it we have the expanding space and on the other side, the field of the radiating star quite uninfluenced by the surrounding expanding space. Since all stars are radiating energy, a simple appeal to the jump conditions of O' Brien and Synge leads one to the conclusion that the expansion of space *does* influence the structure of fields surrounding individual stars.

K. P. SINGH, Banaras : *Coefficients of rotation and gravitational metrics*

The significance of Ricci's twenty-four coefficients of rotation in classifying gravitational spaces has been brought out here. Conditions in terms of $g_{\mu\nu}$ have also been obtained in order that the Riemannian metric of a weak gravitational field be reducible to the orthogonal form (for which half the coefficients of rotation vanish). It has been shown that the line-element of a two-body field cannot be reduced to the orthogonal form on account of the inevitable gravitational motion. The gravitational significance of the geodesic form of the line-element (for which the other half of the coefficients of rotation vanish) is investigated.

P. L. BHATNAGAR and PYARELAL, Delhi : *Shock relations for a Fermi-Dirac gas*

The study of shock waves is becoming more and more prominent in the astrophysical context. When an element of mass crosses a shock wave, discontinuous changes are produced in its physical and dynamical variables. The Rankine and Hugoniot relations determine the jumps in these variables for a perfect gas. The matter of

which stars are composed ranges from perfect gas to degenerate gas. Hence the relations—corresponding to Rankine-Hugoniot relations for a perfect gas—have been obtained to determine the jumps in the physical and dynamical variables for a Fermi-Dirac gas. The perfect gas and the degenerate gas form the two limiting cases of a Fermi-Dirac gas. For these two limiting cases these jumps have been obtained in an explicit form as functions of Mach number of the incoming flow, while for the intermediate range a numerical method is indicated for determining them.

A. S. Apte, Poona : *Agitation in harbours and analytical continuation of functions*

The theory of linear oscillations of the surface of water inside harbours under the action of waves coming from the sea outside has been studied on a model and experimental results are compared with those obtained theoretically. We reduce the physical problem to a Neumann problem for a plane membrane covering a domain occupied by the harbour and the effective portion of the sea. The problem is resolved by means of an auxiliary function of a single variable defining the normal derivative of the potential function along the entrance to the harbour. The analytical continuation of the solution to Neumann's problem defined in the domain of the harbour over that of the sea is possible by means of this function. One is, however, led to an infinite system of linear algebraic equations. Approximate solutions are obtained, which justify, though not rigourously, the experimental and numerical methods. The methods employed appear to be useful in other fields of mathematical physics.

R. Ballabh, Lucknow : *Diffraction of oblique shock*

In 1949 Lighthill investigated mathematically the encounter of a plane normal shock with a solid corner. In an actual problem the incident shock will in general be oblique. In an unpublished paper R. S. Srivastava and myself have proved that after the oblique shock configuration has passed the corner, the region between the incident and reflected shocks will not be disturbed for

all possible shock strengths. In the present paper pressure rise across the reflected shock has been investigated and a mathematical expression obtained for the same.

Denoting $\partial p/\partial y_1 + i\, \partial p/\partial x_1$ by $w(z_1)$ where p denotes the pressure rise multiplied by certain constants, we get

$$\log \frac{w(z_1)}{c} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \text{arc tan} \left(\frac{\partial p/\partial x_1}{\partial p/\partial y_1} \right)_t (t - z_1)^{-1}\, dt,$$

where c is a real constant to be determined from an infinite integral, and the numerator in the above integrand is taken for points of real z_1-axis with x_1 replaced by t.

J. N. Kapur, Delhi : *General theory of moderated charges*

Each grain of a 'moderated' propellant charge consists of a number of layers. The first layer is a cool slow-burning type and the succeeding layers are made of progressively faster burning propellants. The advantage is that as the charge burns the surface area producing the propellant gases may decrease, but as the rate-of-burning constant goes on increasing from layer to layer, even a degressive shape like a chord can behave as a progressive shape and at the same time a higher density of loading can be attained. Corner (1950) pointed out that it should be possible to develop a general theory of moderated charges parallel to the general theory of composite charges. In the present paper, the parallelism has been examined and it has been shown that the parallelism can break down except under certain conditions. When these conditions are satisfied, the form-function for the equivalent charge is obtained and it is shown that in general it is discontinuous at the end of a stage. For degressive surfaces, it is shown that a pressure minimum cannot arise during a stage but can occur in passing from one stage of burning to another. The case when the moderated charge consists of only two layers has been discussed in detail.

M. K. Singal and P. L. Bhatnagar, Delhi: *Moment of inertia of a hypersolid in a Euclidean space E_n of n-dimensions*

In the present paper, the concept of moment of inertia has been generalized to a Euclidean space E_n of n dimensions. Moments

and products of inertia with respect to hyperplanes (and straight lines) in E_n have been defined, and the theorems of parallel and perpendicular axes for hyperplanes (and straight lines) have been established. It has been shown that the $n(n+1)/2$ moments and products of inertia of a hypersolid in E_n with respect to n mutually perpendicular hyperplanes (or straight lines) form the components of a symmetric covariant tensor of the second order. Using Riemannian co-ordinates, the equations of the momental hyperellipsoid, hyperellipsoid of gyration, Legendre's hyperellipsoid and equimomental hypercone have been obtained, and their properties have been studied. A system of particles equimomental with a hyperellipsoid in E_n has been obtained. As expected, the application of the tensor methods has made the treatment more elegant than the corresponding usual treatment for 2 and 3 dimensions. The known results in 2 and 3 dimensions come out as special cases of the results of the present paper.

A. Narasinga Rao and N. S. Subhanarayanan, Chromepet : *On the motion of a perfect fluid near a stagnation point*

GEOMETRY

S. C. Saxena and Ram Behari, Delhi : *Congruences in generalized Riemann space*

Congruences and orthogonal ennuples in generalized Riemann space have been studied. Conditions for a geodesic congruence and a normal congruence have been obtained.

S. C. Saxena and Ram Behari, Delhi : *Hypersurfaces of generalized Riemann space*

The concepts of the angle between two vectors, the magnitude of a vector, and the curvature of a curve in generalized Riemann

space have been introduced. The equations of Gauss and Codazzi have been deduced.

The relation between the curvatures of a curve in a hypersurface and enveloping space has been obtained and a new concept of intrinsic curvature of a curve in a hypersurface of this space has been introduced.

GITA HALDAR and RAM BEHARI, Delhi : *Some properties of automorphic equivalents of vectors in a Kaehler hypersurface*

Defining automorphic equivalence and the angle between two vectors in a Kaehler space, we have studied a complex hypersurface of a Kaehler manifold. In analogy with Riemannian spaces, conjugate directions and lines of curvature have also been defined and studied in relation to their automorphic equivalents.

GITA HALDAR and RAM BEHARI, Delhi : *Orthogonal ennuples in a Kaehler manifold*

An orthogonal ennuple of a Kaehler manifold has been defined and various properties of Ricci coefficients in a Kaehler manifold have been obtained. The necessary and sufficient condition that a system of n mutually orthogonal vectors and their automorphic equivalents, forming an orthogonal ennuple in a Kaehler manifold, form normal congruences of curves has been obtained. It has been shown that under an automorphism a geodesic is preserved if and only if the vector field defining the geodesic is complex analytic.

KAMLA DEVI SINGH, Delhi : *On generalized Riemann space*

A space with coordinates x^i $(i = 1, 2, \ldots, n)$ with which is associated a non-symmetric tensor g_{ij} has been termed by Eisenhart as a generalized Riemann space. The connections for the space are defined by the relation

$$\Gamma k_{ij} = g^{kp} \left(g_{ip,j} + g_{pj,i} - g_{ij,p} \right),$$

where $g^{kp} = \frac{1}{2} \left(g^{kp} + g^{pk} \right)$ and comma (,) followed by an index denotes ordinary derivative. The earlier part of this paper deals with curvature and Ricci tensor and Ricci and Bianchi identities

and in the later part the concepts of parallelism, geodesics, Ricci's coefficients of rotation and Lie derivatives are generalized for this space.

R. S. MISHRA, Delhi : *Non-holonomic subspaces*

In this paper covariant Pfaffian (*Pf*) derivative of a non-holonomic (*NH*) vector has been defined and then the *NH*-curvature tensor, and *NH*-Ricci tensor have been introduced. Mainardi-Codazzi equations of *NH*-hypersurface and *NH*-subspace have been obtained and generalized for a set of *NH*-congruences of curves.

M. K. SINGAL and RAM BEHARI, Delhi : *Riemannian curvature of a hypersurface V_n imbedded in a Riemannian V_{n+1}*

Failkow's theorem (1938) on the difference between the Riemannian curvatures of a V_{n+1} and of a hypersurface V_n imbedded in V_{n+1} states that the difference of the Riemannian curvatures of the V_n and the V_{n+1} for the directions of two lines of curvature at any point, neither of which is tangent to a null vector, is equal to the product of the corresponding normal curvatures.

In the present paper the Riemannian curvatures of a V_{n+1} and of a hypersurface V_n imbedded in V_{n+1} have been considered and Failkow's theorem has been extended for the orientation determined by any pair of conjugate directions. Generalizations of several other theorems and formulae relating to the curvature of a hypersurface—including Gauss's theorem on the total curvature of a surface—have been obtained and several known results have been deduced as particular cases.

M. K. SINGAL and RAM BEHARI, Delhi : *Generalization of normal curvature of a curve in a Riemannian V_n*

In the present paper a congruence of curves through points of a hypersurface V_n imbedded in a Riemannian V_{n+1} has been considered. In analogy with the normal curvature of a curve C in V_n, the generalized normal curvature of C at any point of it, relative to the curve of the congruence through that point, has

been defined as the negative of the resolved part along C, of the derived vector of the unit tangent to the curve of the congruence through the point in the direction of C. The concepts of normal curvature of a hypersurface, principal directions, principal curvatures, lines of curvature, conjugate directions, asymptotic directions and asymptotic lines have been generalized and generalizations of several known theorems on the curvature of a hypersurface V_n in V_{n+1} have been obtained.

R. N. KAUL, Delhi : *Formulae corresponding to Frenet's formulae*

The object of this paper is to consider formulae corresponding to the Frenet's formulae in a Riemannian space of n dimensions.

Let N^α $(\alpha = 1, 2, \ldots, n+1)$ be the unit normal to the hypersurface V_n. We shall denote by $\delta N/\delta S$ the derived vector of the unit normal in the direction of the tangent vector to a curve in the hypersurface. If K_1 is its magnitude and T_1 the unit tangent vector in the direction of this derived vector, we have $\delta N/\delta S = K_1 T_1$. Similarly considering the derived vector of T_1 and resolving it into two components one in the direction perpendicular to $N_1 T_1$ to be denoted by $K_2 T_2$ and the other expressible in terms of N, T_1 we have $-\delta T_1/\delta S = K_2 T_2 - K_1 N$. In this way we arrive at an orthogonal ennuple of $n+1$ vectors $N, T_1, \ldots, T_n$ which is connected in the general form by means of the formulae

$$\frac{\delta T_m}{\delta S} = K_{m+1} T_{m+1} - K_m T_{m-1},$$

with the condition $K_0 = K_{n+1} = 0$.

Various properties of the intrinsic derivatives of these $n+1$ vectors along a curve in the hypersurface have been derived.

M. K. SINGAL and RAM BEHARI, Delhi : *First curvature of a union curve in a hypersurface of a Riemannian space*

A curve on a surface in a Euclidean space of three dimensions is a *union curve* relative to a congruence of curves through points of the surface, its osculating plane at each point contains the ray

of the congruence through that point. Union curves in a hypersurface V_n of a Riemannian V_{n+1} have been studied by C. E. Springer (1950) and T. K. Pan (1954). In the present paper, the differential equations of a union curve in a hypersurface have been obtained in a simpler way and several new expressions have been obtained for its first curvature.

M. SRIRANGAMMA, Bangalore : *Systems of points on a twisted cubic*

A few properties of the twisted cubic, believed to be new, are obtained in this paper. We enunciate some of them here. We take the cubic as $x : y : z : w = t^3 : t^2 : t : 1$.

(1) If two triads ABC, PQR of points on the curve are apolar and if L, M are the poles of the planes ABC, PQR w.r.t. the linear complex of tangents, the line LM generates another linear complex, the pitches of the two complexes being reciprocals of each other.

(2) If $P_1P_2P_3$, $Q_1Q_2Q_3$ are mutually apolar triads the line of instersection of their planes is a bisecant of the cubic.

(3) If ABC, $A'B'C'$ are cubicovariant triads of points and O, O' their Hessian points, the planes OAA', OBB', OCC' intersect along the line OP where P is the point where the tangent at O meets the osculating plane at O'.

(4) If the polar plane of A w.r.t. the tetrad $ABCD$ and the plane BCD are conjugate w.r.t. the linear complex, then A, B, C, D form an equianharmonic tetrad.

M. N. RAJALAKSHAMMA, Bangalore : *The reciprocal of a linear complex*

If C is a linear complex of lines, the polars of C w.r.t. another linear complex Γ form a linear complex C'. The relation between C and C' is reversible. We call C' as the reciprocal of C w.r.t. Γ. C and C' have the same pitch, and they coincide if C and Γ are mutually apolar.

Two mutually apolar complexes C_1 and C_2 reciprocate into apolar complexes C_1' and C_2'. The common transversals of a linear congruence reciprocate into the transversals of the reciprocal congruence. In particular, a parabolic congruence reciprocates into another parabolic congruence.

K. N. KAMALAMMA, Bangalore : *Congruence of tangents to a system of curves on a surface*

(a) If the curves are a family of geodesic parallels, it is proved that the two ruled surfaces whose lines of striction lie on the surface of reference are at right angles.

(b) If a system of curves be lines of curvature as well as geodesic parallels, the characteristic ruled surfaces of the congruence of tangents coincide with the surfaces whose spherical representations are minimal lines.

Considering the two congruences formed by tangents to both systems of lines of curvature on any surface, we prove the following :—

(1) The product of the distances of the four limit points at a point on $S = 1/16K^2$ where K is the Gaussian curvature.

(2) The sum of the parameters of distribution of the characteristic ruled surfaces of the two congruences is equal to $2J/K \tan\phi \tan\phi'$ where ϕ and ϕ' are the angles between the focal planes of the two congruences and J is the mean curvature.

(3) The ratio of the focal distances of the two congruences is equal to the tangent of the angle between the two lines of striction.

(4) For the congruence of tangents to a system of asymptotic lines, one of the mean ruled surfaces is a developable, while the parameter of distribution of the other is equal to the radius of torsion of the asymptotic line.

C. N. SRINIVASIENGAR and H. S. NAGARATHNAMMA, Bangalore : *Inversion of a hypersurface in Riemannian space*

Let V_{n+1} be a Riemannian space with $a_{\alpha\beta}$ as its fundamental tensor and y^α, $(\alpha = 1, \ldots, n+1)$ the coordinates of any point.

If V_n is a hypersurface of V_{n+1}, we apply to V_n the transformation $\xi^\delta = (c^2 y^\delta)/(a_{\alpha\beta} y^\alpha y^\beta)$, $\delta = 1, \ldots, n+1$. V_n is transformed into Q_n, and we easily see that the inverse transformation is of the same form, viz., $y^\delta = (c^2 \xi^\delta)/(a_{\alpha\beta} \xi^\alpha \xi^\beta)$. This gives the analogue of the process of inversion in ordinary Euclidean geometry. We prove the following theorems whose analogues are well known in Euclidean space.

(1) If $\overline{N}^\alpha$ and $\overline{N}^\alpha$ are the components of the unit normal to V_n and Q_n at corresponding points, $\overline{\overline{N}}^\alpha = (2p/\psi^2)\, y^\alpha - N^\alpha$, where $p = a_{\alpha\beta} y^\alpha N^\beta$ and $\psi^2 = a_{\alpha\beta} y^\alpha y^\beta$. Hence two hypersurfaces intersect at the same angle as their transforms.

(2) Lines of curvature of V_n transform into lines of curvature of Q_n.

(3) The normal curvatures of V_n and Q_n are connected by the relation $\overline{\kappa} = -\psi^2\kappa/c^2 - 2p/c^2$.

These results are further generalized by the transformation $\xi^\alpha = \phi y^\alpha$ where ϕ is a function of the coordinates of V_n. This is the generalization of what Weatherburn calls "conical projection" (*Differential geometry*, Vol. II).

KAMALA DEVI SINGH, Delhi : *Subspaces of semi-simple group-spaces*

P. S. RAU, Tirupati : *On the existence theorems of the conic and the conicoid*

K. S. SEN, Tirupati : *On eccentric lines of an ellipse*

STATISTICS

V. S. Huzurbazar, Poona : *On successive approximations to the maximum likelihood estimator*

In this paper attention is drawn to the use of transformations in successive approximations to the maximum likelihood estimator.

P. C. Consul, Khurja, U.P. : *On the distribution of the ratio* $D_1'^2/D_2'^2$ *of the divergence* D' *between two multinomial populations*

A. Bhattacharya (*Sankhya*, 7(4), 1946) has defined a new measure of divergence D' between two samples of sizes n_1 and n_2 drawn from two multinominal populations, and obtained the distribution of D'. If D_1' be the Bhattacharya divergence for a pair of samples of sizes n_1 and n_2, and D_2' that for another pair of the same sizes, we define a new variable W by $e^W = D_1'^2/D_2'^2$ and study its distribution.

K. S. V. Shenoi, Trivandrum : *From perfect competition to successive monopoly*

G. Sankaranarayanan, Annamalainagar : *Distributions of the sum of the products two at a time of a set of independent random variables*

K. S. Sen, Tirupati : *The power function of the Chi-square test*

N. V. Dave, Junagadh : *Statistical study of the wholesale prices in India after independence*

M. V. Jambunathan, Mysore : *An analysis of the answers to a typical intermediate question paper by students of the Mysore University*

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NEWS AND NOTICES

The following persons have been admitted to life-membership in the Society : P. D. Gupta, J. N. Kapur, V. S. Krishnan, S. R. Nagpaul, Shanti Narayan, M. S. Ramanujan.

The following persons have been admitted to membership in the Society : S. V. Adavi, M. C. Agarwala, L. D. Chatterjee, M. S. Cheema, V. P. Dang, Bansari Das, B. Dube, J. Dutt, N. D. Gautam, M. C. Gupta, P. M. Gupta, R. N. Gupta, R. P. Guru, M. S. Huzurbazar, R. Jha, N. A. Khan, M. L. Madan, P. N. Maini, M. Misra, S. Moran, K. S. Nagaraja, K. R. Nagarajan, V. C. Nanda, S. Ramakrishnan, M. S. V. Raman, M. Ranganathan, P. V. Ranganathan, Shyamala Rau, A. G. Singaravelu, Labh Singh, Rajendra Singh, Rattan Singh, U. N. Singh, Srinivasa Rangachari, S. Swetaranyam, V. S. Thawani, R. Venkataraman, P. D. Verma, E. P. Zachariah.

The following members of the American Mathematical Society have been admitted as members of the Society under the reciprocity agreement : John D. Baum, S. G. Bowne, R. C. Buck, G. A. Hedlund, N. D. Kazarinoff, J. C. Oxtoby, Henry O. Pollak, A. L. Shields, James Singer, W. R. Utz.

The Commission on Scientific Publications of the International Mathematical Union consists of B. A. Amira, R. P. Boas, W. Fenchel and G. Sansone. The commission would be glad to receive suggestions from mathematicians about what problems it should study and what it might hope to accomplish. Suggestions may be sent to any of the members of the commission, but should preferably be addressed to the chairman, R. P. Boas, Lunt Building, Northwestern University, Evanston, Illinois, U.S.A.

The First Indian Congress on Theoretical and Applied Mechanics was held on November 1-2-1955 at the Indian Institute of Technology, Kharagpur, under the presidentship of Professor K. S. Krishnan. The Indian Society of Theoretical and Applied Mechanics was constituted at an *ad hoc* meeting of the Congress held on November 1. The following office-bearers were elected :

Professor K. S. Krishnan (President), Dr. V. M. Ghatage, Professor N. R. Sen (Vice-Presidents), Professor B. R. Seth (Secretary-Treasurer), Dr. C. V. Joga Rao, Dr. C. P. Chatterji, Professor B. M. Belgaumkar, Dr. S. K. Roy, Dr. Ram Ballabh, Dr. V. Cadambe, Dr. P. L. Bhatnagar, Sri K. L. Rao and Professor R. Khanna (Members). Those interested in becoming members may contact the Secretary.